Unprovability and Beyond

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Hilbert's Program

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- Establish Axioms to prove all mathematical truth;
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Gödel's Answer

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- In fact, no consistent axiom system can prove its own consistency, i.e., the statement that the system is consistent.

A combinatorial result:

Notation

We use $[X]^k$ to denote the collection of all k-element subsets of X, and $N = \{0, 1, 2 \dots, N-1\}$.

Ramsey's Theorem

For every k, m, there is a number N such that for any $f : [N]^k \to \{0, 1\}$, there is a subset $X \subset N$ of size at least m such that $f \restriction [X]^k$ is constant.

Definition

A finite set of natural numbers is large if its size is greater than or equal to its least element.

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Paris-Harrington Theorem

The Modified Ramsey's Theorem is not provable in Peano Arithmetic (PA, basic axioms about natural numbers plus induction).

Quantifier Complexity

Given a sentence P in arithmetic $(\mathbb{N}, +, \times)$, we define the quantifier complexity by the number of alternations of quantifiers ($\forall$ and $\exists$) in P.

Examples:

- Goldbach Conjecture (Π_1)
- Odd Perfect Number Conjecture (Π_1)
- Twin Prime Conjecture (Π_2)
- "$3x + 1$" Conjecture (Π_2)
- "P $\neq$ NP" (Π_2)
- Riemann Hypothesis (equivalent to Π_1)
- $\pi + e$ is rational (Σ_2)
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- "X is provable/unprovable in PA" (Σ_1/Π_1)

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- "P is unprovable" is a Π_1 sentence.
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- "P is unprovable" is a Π_1 sentence.
- This sentence proves the con(PA), the consistency of PA.
- For the Gödel's self-reference sentence, or the Modified Ramsey Theorem, we only need con(PA) to prove that it is unprovable.
- So is Con(PA) always enough to prove such unprovability statement?

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For every true Π_1 sentence Q which proves the con(PA), there is a true sentence P such that PA proves:

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A sentence Q is provably equivalent to the unprovability of some sentence if and only if Q is provably equivalent to a Π_1 sentence which proves the con(PA). So the unprovability of a sentence could be arbitrarily hard to prove.

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- If we detect that $Prov(P)$ is true, i.e., we find a proof from PA to P,then we make P provably equivalent to $Con(PA)$. (So if this happens, then PA proves $Con(PA)$, and by Gödel, this implies $\neg Con(PA)$, which implies $\neg Q$).

Theorem [C.]

Given any consistent theory T extending the base theory PA, we can always find a true sentence P such that:

- T does not prove P, and
- T does not prove "P is unprovable in PA".

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- So how about proving everything from true Π_1 sentences?

The story continues...

The Modified Ramsey Theorem (which is Π_2) is not provable in PA from any true Π_1 sentence.

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Relativize the previous proof above $\mathbf{0}^{(n-1)}$.

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Gödel:	Well, when you wake up from one dream, it is possible that you are still dreaming.