On Absoluteness of Mathematical Truth

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Definiteness of a structure implies the definiteness of its truth

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Another perspective

- Models of set theory can have the same structure of arithmetic $(\mathbb{N}, +, \cdot, 0, 1)$, yet disagree on arithmetic truth.
- Models of set theory can have the same reals, yet disagree on projective truth.
- Models of set theory can have a transitive rank initial segment V_δ in common, yet disagree about its truth.

These suggest that the determinateness of truth for a structure is also a higher-order commitment.

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Models of set theory may have different arithmetic truth, projective truth, set theory truth, etc. In most cases, we prove these independence results buy adding something in or cutting something off.

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Main theorem

Every consistent extension of ZFC has two models M_1 and M_2, which agree on the natural numbers and on the structure $(\mathbb{N}, +, \cdot, 0, 1)^{M_1} = (\mathbb{N}, +, \cdot, 0, 1)^{M_2}$, but which disagree on arithmetic truth, in the sense that there is $\sigma \in \mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ such that both M_1 and M_2 think σ is an arithmetic sentence, while M_1 thinks σ is true, but M_2 thinks it is false.

The proof

- We start with a computably saturated countable model M of ZFC and its arithmetic structure $(\mathfrak{N}, \mathrm{Sa})^M$ with the satisfaction class or truth class Sa, which is ZFC-definable.
- Since Sa is not definable in $\mathfrak{N}$, there exists $\sigma, \tau \in \mathbb{N}^M$ realizing the same 1-type in $\mathfrak{N}^M$, such that $\sigma \in \mathrm{Sa}^M$ but $\tau \notin \mathrm{Sa}^M$.

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The proof (continued)

- Too see such σ and τ exists, consider the computable 2-type

$$p(s,t) = \{\varphi(s) \leftrightarrow \varphi(t)\}_{\varphi \in L_{Ar}} \cup \{s \in \mathsf{Sa} \wedge t \notin \mathsf{Sa}\}.$$

It is finitely realized simply because Sa^M is not definable in $\mathfrak{N}^M$.

The proof (continued.)

- Now we have arithmetics sentences σ and τ realizing the same 1-type in $\mathfrak{N}^M$, yet M thinks σ is true and τ is false in $\mathfrak{N}^M$.
- By a back and forth construction, there is an automorphism π on $\mathbb{N}^M$ with $\pi(\tau) = \sigma$.

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The proof (continued..)

- Extend π to be an isomorphism $\pi^* : M \to M'$.
- Thus an element $m \in \mathbb{N}^M$ sits inside M the same way that $\pi(m)$ sits inside M'.
- The situation is that M and M' has the same natural numbers, and M thinks σ is truth, but M' thinks σ is false because $\sigma = \pi(\tau)$ and M thinks τ is false.

  [QED]

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Some remarks

- In the proof, we only need the structure to be able to handle Gödel's coding and so the satisfaction class is an undefinable subclass of the structure. Therefore, the theorem can be easily generalized.
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Disagree on a “readable” sentence

Theorem

Every countable model M of ZFC has elementary extensions M_1 and M_2, with a transitive rank-initial segment $(V_\delta, \in)^{M_1} = (V_\delta, \in)^{M_2}$ in common, yet M_1 thinks V_δ violates Σ_n-collection firstly on an even number n, and M_2 thinks it is odd.

Sketch of proof

- Begin with a countable ZFC model M, take

$$T_1 = \Delta(M) + \text{“}V_\delta \prec V\text{''}$$
$$+ \text{“the least } n \text{ such that } V_\delta \nprec \Sigma_n\text{-collection is even''}$$

  and T_2 correspondingly.

- Both T_1 and T_2 are consistent, take computably saturated model pair of T_1 and T_2, and get the isomorphism.

Some philosophical remarks

Coordinate system to measure one's view on the philosophy of mathematics

- realism in ontology vs. idealism (nominalism in ontology)
- realism in truth-value vs. anti-realism truth-value

Some philosophical remarks

There is a prima facie alliance between realism in truth-value and realism in ontology... [Although] The ontology thesis that numbers exists objectively may not directly follow from the semantic thesis of truth-value realism

Stewart Shapiro

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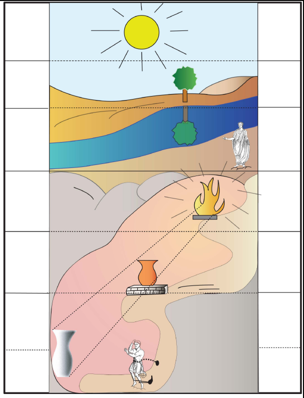

Figure: Plato's allegory of the Cave

Thank You

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