On Strong Π^1_1-Martin-Löf Randomness

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Spector-Gandy Theorem

Theorem (Spector; Gandy)

A real $x \subseteq \omega$ is Π^1_1 if and only if it is Σ_1 over $L_{\omega_1^{\mathrm{CK}}}$

Based on Spector-Gandy Theorem, we may define a Turing reduction in the higher setting.

Definition (Greenberg, Laurent and Monin)

A real $x \leq_{hT} y$ if there is a $\Sigma_1(L_{\omega_1^{\mathrm{CK}}})$ relation $\Phi \subseteq 2^{<\omega} \times 2^{<\omega}$ so that

1. *If τ_0 and τ_1 are comparable, then for any σ_0, σ_1, if $\Phi(\sigma_0, \tau_0)$ and $\Phi(\sigma_1, \tau_1)$, then σ_0 and σ_1 are comparable; and*
2. *For any $\sigma \prec x$, there is some $\tau \prec y$ such that $\Phi(\sigma, \tau)$.*

Martin-Löf Randomness

Definition (Martin-Löf)

(i) *A Martin-Löf test is a computable collection* $\{V_n : n \in \mathbb{N}\}$ *of c.e. sets such that* $\mu(V_n) \leq 2^{-n}$.

(ii) *A real* y *is said to pass the Martin-Löf test if* $y \notin \bigcap_{n\in\omega} V_n$.

(iii) *A real* y *is Martin-Löf random if it passes all the Martin-Löf tests.*

Π^1_1-Martin-Löf Randomness

Definition (Hjorth and Nies)

(i) *A Martin-Löf test is a* Π^1_1 *collection* $\{V_n : n \in \mathbb{N}\}$ *of* Π^1_1*-coded open sets such that* $\mu(V_n) \leq 2^{-n}$.

(ii) *A real* y *is said to pass the* Π^1_1 *Martin-Löf test if* $y \notin \bigcap_{n\in\omega} V_n$.

(iii) *A real* y *is* Π^1_1*-Martin-Löf random if it passes all the* Π^1_1 *Martin-Löf tests.*

The collection of Π^1_1-Martin-Löf random reals is $\mathbf{\Sigma}^0_2$ but not $\mathbf{\Pi^0_2}$.

A universal Π^1_1-Martin-Löf Random Real

There is a Π^1_1-Martin-Löf random, left-Π^1_1- real $\Omega_{\Pi^1_1}$ so that for any left Π^1_1 real x, $x \leq_{hT} \Omega_{\Pi^1_1}$.

Strong ML-Randomness

Definition (Kurtz)

(i) *A generalized Martin-Löf test is a computable collection* $\{V_n : n \in \mathbb{N}\}$ *of c.e. sets such that* $\lim_{n\to\infty} \mu(V_n) = 0$.

(ii) *A real* y *is said to pass the generalized Martin-Löf test if* $y \notin \bigcap_{n\in\omega} V_n$.

(iii) *A real* y *is strong ML-random if it passes all the generalized Martin-Löf tests.*

Π^1_1-strong ML-Randomness

Definition (Nies)

(i) *A generalized Π^1_1 Martin-Löf test is a Π^1_1 collection $\{V_n : n \in \mathbb{N}\}$ of Π^1_1 coded open sets such that* $\lim_{n\to\infty} \mu(V_n) = 0$.

(ii) *A real y is said to pass the generalized Π^1_1 Martin-Löf test if $y \notin \bigcap_{n\in\omega} V_n$.*

(iii) *A real y is Π^1_1 strong ML-random if it passes all the Π^1_1-generalized Martin-Löf tests.*

Separating Randomness Notions

Theorem (Yu; Greenberg, Laurent and Monin)

Π^1_1 *strong ML-randomness is properly stronger than* Π^1_1-*ML-randomness. Actually* $\Omega_{\Pi^1_1}$ *is not strong ML-randomn.*

Proof.

We use an argument due to GLM. □

A technique lemma

Lemma

Suppose that $\{U_n\}_{n\in\omega}$ *is a uniformly* Π^1_1*-sequence of open sets. If there is a* $\Sigma_1(L_{\omega_1^{\mathrm{CK}}})$ *enumeration* $\{\hat{U}_{n,\gamma}\}_{n\in\omega,\gamma<\omega_1^{\mathrm{CK}}}$ *of the sequence with two numbers* k *and* $m \geq 1$ *such that for every* n*,* $U_n = \bigcup_{\gamma<\omega_1^{\mathrm{CK}}} \hat{U}_{n,\gamma}$ *and for every* $\gamma < \omega_1^{\mathrm{CK}}$*:*

(a) $\hat{U}_{n+1,\gamma} \subseteq \hat{U}_{n,\gamma}$ *and each string in* $\hat{U}_n$ *has length at least* $2^{k\cdot n}$*,*

(b) $\forall\sigma \in 2^{k\cdot n-m}(\mu(\hat{U}_{n,\gamma} \cap [\sigma]) < 2^{-1+m-k\cdot n})$*, and*

(c) *For* $\gamma < \omega_1^{\mathrm{CK}}$ *and any real* z*, if* $z \in \hat{U}_{n,<\gamma} \setminus \hat{U}_{n,\gamma}$*, where* $\hat{U}_{n,<\gamma} = \bigcup_{\beta<\gamma} \hat{U}_{n,\beta}$*, then* $z \notin \hat{U}_{n,\beta}$ *for any* $\beta \geq \gamma$*.*

Then $\{U_n\}_{n\in\omega}$ *is a generalized* Π^1_1*-ML-test.*

Characterizing the difference via hyperarithmetic reduction

Theorem (Chong and Yu)

For any hyperdegree $\mathbf{a}$, $\mathbf{a}$ *contains a* Π^1_1*-ML-random but not strong* Π^1_1*-ML-random real if and only if* $\mathbf{a} \geq_h \mathbf{0}'$.

Proof.

Combining the Lemma with Kučera coding. $\square$

Π^1_1-Difference Randomness

Definition

(i) *A d-Π^1_1 test is a Π^1_1 collection $\{V_n : n \in \mathbb{N}\}$ of Π^1_1 open sets and a Σ^1_1-closed-set T such that $\forall n \mu(V_n \cap T) < 2^{-n}$.*

(ii) *A real y is said to pass the d-Π^1_1 test if $y \notin \bigcap_{n\in\omega} V_n \cap T$.*

(iii) *A real y is Π^1_1 difference random if it passes all the-Π^1_1 tests.*

Characterizing Π^1_1-Difference-Randomness via hT-reduction

Theorem (Nies)

A Π^1_1-ML-random real x is not Π^1_1 difference random if and only if $\Omega_{\Pi^1_1} \leq_{hT} x$.

Proof.

$\leftarrow$. A routine argument from classical randomness due to Levin-Miller-Yu.
$\rightarrow$. Fix a Σ^1_1-closed set T and a Π^1_1 collection $\{V_n : n \in \mathbb{N}\}$ of Π^1_1 sets such that $x \in \bigcap_{n\in\omega}(T \cap V_n)$. For any k, if $m \geq k$ belongs to $\mathscr{O}$ at some stage β and $\mu(T[\gamma] \cap V_m[\beta]) \leq 2^{-k}$ for some $\gamma \geq \beta$, we put an open set with measure $\leq 2^{-k+1}$ covering the difference test at stage γ into U_k. Then $\{\bigcup_{k\geq n+1} U_k\}_{n\in\omega}$ is a Π^1_1-ML-test and $x \notin \bigcap_n \bigcup_{k\geq n+1} U_k$. Then $\mathscr{O} \leq_{hT} x$. □

Strong Π^1_1-ML-Randomness implies Difference Randomness

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Theorem (Greenberg, Laurent and Monin)

No strong Π^1_1-ML-Randomness is hT above $\Omega_{\Pi^1_1}$. So strong Π^1_1-ML-randomness implies difference randomness.

Proof.

By a routine argument from classical randomness due to Levin-Miller-Yu, it can be shown that if x is strong Π^1_1-ML-Random and $y \leq_h x$, then so is y. Then apply $\Omega_{\Pi^1_1}$. $\square$

The Borel Rank of Strong Π^1_1-ML-Randomness

Theorem (Yu)

The collection of strong Π^1_1-ML-random reals is not $\boldsymbol{\Sigma}^0_3$.

Proof.

By a forcing argument. The key point is to apply Π^1_1-difference tests. $\square$

Thanks!