Classical Theorems in Reverse Recursion and α-Recursion

Li Wei
liwei.sg@gmail.com

Department of Mathematics
National University of Singapore

15 June, 2013

Outline

Reverse Recursion Theory

- Study Recursion Theory “Reversely”.

Reverse Recursion Theory

- Study Recursion Theory "Reversely".
- Main question: How much induction do we need to prove theorems in Recursion Theory?
  V.S. Main question of Reverse Mathematics: What set existence axiom do we need to prove theorems in algebra, analysis, geometry, etc.?

Reverse Recursion Theory

- Study Recursion Theory “Reversely”.
- Main question: How much induction do we need to prove theorems in Recursion Theory?
  V.S. Main question of Reverse Mathematics: What set existence axiom do we need to prove theorems in algebra, analysis, geometry, etc.?
- It was started in 1980s, using first order language.

Reverse Recursion Theory

- Study Recursion Theory "Reversely".
- Main question: How much induction do we need to prove theorems in Recursion Theory?
  V.S. Main question of Reverse Mathematics: What set existence axiom do we need to prove theorems in algebra, analysis, geometry, etc.?
- It was started in 1980s, using first order language.
- The first result, which says Friedberg-Muchnik theorem is provable in Σ_1 induction, was got by S. Simpson.

Hierarchy of Induction (by Pairs and Kirby, 1978; Mytilinaios and Slaman, 1989; Slaman, 2004)

$I\Sigma_2 \Leftrightarrow I\Pi_2$: Infinite injury

$B\Sigma_2 \Leftrightarrow I\Delta_2$

$I\Sigma_1 \Leftrightarrow I\Pi_1$: Finite injury

$\underline{B\Sigma_1} \Leftrightarrow I\Delta_1$

α-Recursion

- In the 1960's, Kreisel suggested the idea of generalizing the syntactic aspects of classical recursion theory, building on the earlier works of Church, Gandy, Kleene, Spector and Kreisel himself.

α-Recursion

- In the 1960's, Kreisel suggested the idea of generalizing the syntactic aspects of classical recursion theory, building on the earlier works of Church, Gandy, Kleene, Spector and Kreisel himself.
- Sacks pursued this idea and developed recursion theory on admissible ordinals.

α-Recursion

- In the 1960's, Kreisel suggested the idea of generalizing the syntactic aspects of classical recursion theory, building on the earlier works of Church, Gandy, Kleene, Spector and Kreisel himself.
- Sacks pursued this idea and developed recursion theory on admissible ordinals.
- Admissible ordinal α: $L_\alpha \models \Sigma_1$ replacement.

α-Recursion

- In general, admissible ordinals lack certain combinatorial properties that come with the standard model ω and crucial to the construction of r.e. sets.

α-Recursion

- In general, admissible ordinals lack certain combinatorial properties that come with the standard model ω and crucial to the construction of r.e. sets.
- This results in constructions which are sometimes much more intricate than those for ω, and in certain cases, the failure of the combinatorial property leads to a negative conclusion.

Motivations

Fragments of Peano Arithmetic and α-Recursion Theory each provides a platform for the study of models of computation to study recursion theory.

Motivations

Fragments of Peano Arithmetic and α-Recursion Theory each provides a platform for the study of models of computation to study recursion theory.

- The key properties of a computation should not depend solely on the underlying structure of the standard model.

Motivations

Fragments of Peano Arithmetic and α-Recursion Theory each provides a platform for the study of models of computation to study recursion theory.

- The key properties of a computation should not depend solely on the underlying structure of the standard model.
- Therefore, it is necessary and possible to consider notions of computation in a more general setting.

They also offer a framework for the investigation of logical strength.

They also offer a framework for the investigation of logical strength.

- Reverse Recursion Theory: investigate the inductive strength required to show theorems in Recursion Theory.

They also offer a framework for the investigation of logical strength.

- Reverse Recursion Theory: investigate the inductive strength required to show theorems in Recursion Theory.
- α-Recursion Theory: investigate the necessary and sufficient replacement axioms for theorems in Recursion Theory.

Reverse Recursion V.S. α-Recursion

	Reverse Recursion	α-Recursion
Language	Language of Arithmetic $\mathcal{L}(0, 1, +, \cdot)$	Language of Set Theory $\mathcal{L}(\in)$
Axioms	P^-, $I\Sigma_n$, $B\Sigma_n$, etc	Σ_1 replacement
Models	Nonstandard models of arithmetic with restricted induction	L_α, where α is admissible
Difficulty	lack of induction	lack of replacement
r.e. sets	Σ_1 definable	
finite sets	have a code in the model	
Turing reducibility	setwise	

Results

	Fragments of PA	α-recursion theory
Friedberg-Muchnik	Every model of $B\Sigma_1$ (Chong and Mourad)	every admissible ordinal (Sacks and Simpson)
Sacks' Splitting	Equivalent to $I\Sigma_1$ over $B\Sigma_1$ (Mytilinaios)	every admissible ordinal (Shore)
Sacks' Density	Every model of $B\Sigma_2$ (Groszek, Mytilinaios and Slaman)	every admissible ordinal (Shore)
Minimal Pair	Equivalent to $I\Sigma_2$ over $B\Sigma_2$ (Chong, Qian, Slaman and Yang)	some admissible ordinals [partially open] (Lerman, Sacks, Shore, Maass)

Example 1 — Shore's blocking method

Theorem 1

- *If α is admissible, then Sacks' splitting theorem holds in L_α.*
- *In reverse recursion theory, Sacks' splitting theorem is true in every model satisfying Σ_1 induction.*

Idea. A straightforward application of the classical proof requires Σ_2 replacement in α-recursion and Σ_2 induction in reverse recursion.

Sacks' Splitting — α-Recursion

Suppose W is a non-recursive r.e. set in L_α and we would like to construct two r.e. sets A, B such that

- $A \cup B = W$ and $A \cap B = \emptyset$.
- $A \not\geq_T W$ and $B \not\geq_T W$.

Requirement:

$$A \neq \Phi_e^W, \qquad B \neq \Phi_e^W.$$

Each requirement is injured *finitely* many times.

Shore's blocking method — α-Recursion

If L_α does not satisfy Σ_2 replacement, then there is the least $\beta < \alpha$ such that

$$\exists f \, (f : \beta \to \alpha \text{ is cofinal and } \Sigma_2 \text{ definable over } L_\alpha)$$

Shore's blocking method — α-Recursion

If L_α does not satisfy Σ_2 replacement, then there is the least $\beta < \alpha$ such that

$$\exists f \, (f : \beta \to \alpha \text{ is cofinal and } \Sigma_2 \text{ definable over } L_\alpha)$$

Consider requirements, $A \neq \Phi_e^W$ for e's in one block as one requirement.

Shore's blocking method — Reverse Recursion

Straightforward extension of the idea in α-Recursion does not work: there is the least $\beta < \alpha$ such that

$$\exists f\,(f : \beta \to \alpha \text{ is cofinal and } \Sigma_2 \text{ definable over } L_\alpha)$$

Modification: blocks are defined dynamically. There are cofinally many blocks.

Example 2 — Friedberg Numbering

Definition 2

Let $\{A_e\}$ be an effective list of r.e. sets. We say $\{A_e\}$ is a Friedberg Numbering if

(i) $e \neq d \rightarrow A_e \neq A_d$.

(ii) For every r.e. set A, there is an e such that $A_e = A$.

In other words, a Friedberg numbering is an effective choice function for the collection of the equivalence classes of r.e. sets under set equality.

Question: Is there a Friedberg numbering for a given model of computation?

Standard Model

Let $\{W_e\}$ be an effective list of all r.e. sets (may have repetitions).
Observation: For every r.e. set W, there is an e such that $W = W_e$ and

$$\forall d < e \, (W_d \neq W_e). \tag{1}$$

Moreover, (1) is equivalent to a Σ_2 statement. Thus, there is a method to recursively approximately determine whether e has property (1).

Fragments of PA without Σ_2 Induction

Suppose $\mathcal{M} \models B\Sigma_2 + \neg I\Sigma_2$.
Question: is the following still true?
"For every r.e. set W, there is an e such that $W = W_e$ and

$$\forall d < e\,(W_d \neq W_e).\text{"}$$

Fragments of PA without Σ_2 Induction

Suppose $\mathcal{M} \models B\Sigma_2 + \neg I\Sigma_2$.
Question: is the following still true?
"For every r.e. set W, there is an e such that $W = W_e$ and

$$\forall d < e\,(W_d \neq W_e).\text{"}$$

Answer: No. In fact, there is no Friedberg numbering for such models.

Fragments of PA without Σ_2 Induction

Suppose $\mathcal{M} \models B\Sigma_2 + \neg I\Sigma_2$.
Question: is the following still true?
"For every r.e. set W, there is an e such that $W = W_e$ and

$$\forall d < e \, (W_d \neq W_e)."$$

Answer: No. In fact, there is no Friedberg numbering for such models.
Strategy: Given a recursive list of r.e. sets $\{A_e\}$ without repetition, construct an r.e. set not in the list.

Lemma 3

For every $a \in \mathcal{M}$, there is a b such that

$$\forall d, e < a \, (d \neq e \rightarrow A_d \upharpoonright b \neq A_e \upharpoonright b).$$

Result

Theorem 4

Over $P^- + B\Sigma_2$, Σ_2 induction holds if and only if there is a Friedberg numbering of all r.e. set.

Note: $I\Sigma_2$ — infinite injury.

Admissible Ordinals

Let α be an admissible ordinal.
If $L_\alpha \models \Sigma_2$ replacement, then there is a Friedberg numbering.
Now we consider the case that Σ_2 replacement fails in L_α.
Observation: For every r.e. set W, there is an e such that $W = W_e$ and

$$\forall d < e\,(W_d \neq W_e). \tag{2}$$

But (2) is Π_3 so does not permit a recursive approximation.
Two representative cases: ω_1^{CK}, $\aleph_\omega^L$.

Case 1. $\alpha = \omega_1^{\mathsf{CK}}$

Lemma 5

There is a $\Sigma_1(L_\alpha)$ definable injection $p : \omega_1^{\mathsf{CK}} \to \omega$.

Therefore, the construction of a Friedberg numbering can be "essentially" carried out over L_ω.

Case 2: $\alpha = \aleph_\omega^L$

Conclusion: There is no Friedberg numbering in L_α.
Strategy: Proof by diagonalization.
Suppose $\{A_e\}_{e<\alpha}$ is a recursive list of r.e. sets without repetition. The objective is to construct an r.e. set X not in the list.

Proposition 6

There is an increasing $\Sigma_2(L_\alpha)$ cofinal function $f : \omega \to \aleph_\omega^L$ such that $f(n) = \aleph_n^L$.

Remark. Unlike the case for $B\Sigma_2$, in general, given an n, there is no b such that

$$\forall d, e < \aleph_n^L \,(d \neq e \,\rightarrow\, A_d \restriction b \neq A_e \restriction b)$$

We use stable ordinals in $\aleph_\omega^L$, coding, and more to get around the difficulty and ensure the diagonalization successes.

Theorem

The characterization of Friedberg numbering for admissible ordinals is stated in terms of fine structure theory:

Theorem 7

$t\sigma 2p(\alpha) = \sigma 2cf(\alpha)$ *if and only if there is a Friedberg numbering.*

Thank you!