How Philosophy Impacts on Mathematics

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Background

The preliminary to contemporary philosophy of mathematics:

- Current status of the research in foundation of mathematics

Axiomatization of mathematics

Model:

- *The Elements*

The way to rigorousness

- The formalization of mathematical language
- The formalization of mathematical deduction
- The choose of axioms

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Some Typical Axiomatic Systems

- First order arithmetic:
  Robinson arithmetic, Peano arithmetic
- Second order arithmetic:
  PRA_0, RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0, etc.
- Axiomatic set theory:
  ZF, ZFC, large cardinal axioms, forcing axioms, etc.

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Completeness

Definition (Completeness)

A theory T (set of formulas) is complete if and only if for each formula (in the same language of T) φ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

None of the above axiomatic systems is complete. Moreover,

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Independent statements

- Let T be an consistent axiomatic system, then Con(T) is independent from T.
- The continuum hypothesis (CH) is independent from ZFC, and even ZFC plus large cardinal axioms.
- The axiom of projective determinacy (PD) is independent from ZFC.

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The phenomenon of incompleteness

Today, the brands of different philosophies of mathematics are their attitudes towards the phenomenon of incompleteness.

Consistency strength

Let T_1, T_2 be theories, the strict order of consistency strength is defined as follow.

$$T_1 < T_2 \quad \text{if and only if} \quad T_2 \vdash \text{Con}(T_1).$$

We say T_1 and T_2 are equiconsistent (based on theory T_0) if,

$$T_0 \vdash \text{Con}(T_1) \leftrightarrow \text{Con}(T_2).$$

Gödel Hierarchy

. . . ZFC . . .

. . . type theory

Π^1_1-CA_0 (Π^1_1 comprehension)

ATR_0 (arithmetical transfinite recursion)

ACA_0 (arithmetical comprehension)

WKL_0 (weak König's lemma)

. . . RCA_0 (recursive comprehension)

Q (Robinson arithmetic)

Gödel hierarchy extended (large cardinals)

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. . . *n*-Huge

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Irrelevancies

I am ruling out what are not perfectly fitting with the topic

Philosophically neutral mathematics

- No one would have any doubt on $5 + 7 = 12$
  no matter you are realist or nominalist, rationalist or empiricist
- No one would have any doubt on the axioms of Robinson arithmetic (Q), e.g.,

$$x + Sy = S(x + y)$$

$*$ Although some strict finitists may have very restrictive interpretation on the range of the arguments in the equation.

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Mathematically neutral philosophy

- Many delicate philosophically differences are not acknowledged by the community of mathematicians.
  e.g. some epistemologically differences between realists.
  We are temporarily not interested in such differences.
- Some philosophical ideas are arguably equivalent concerning their impacts on mathematical practice.
  I have argued that the set theory multiverse view is practically either compatible with the traditional set theory realism or equivalent to ZFC formalism.

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Intuitionism

Brouwer:

> *A mathematical statement corresponds to a mental construction, and a mathematician can assert the truth of a statement only by verifying the validity of that construction by intuition.*

Thus the principle of excluded middle is not valid.

The Failure of Revisionism

Technically: All results in classical mathematics can be acknowledged by an intuitionist

- Gödel-Gentzen negative translation
- Gödel's coding

The Failure of Revisionism

Practically: The influence of intuitionism is fading away among the community of mathematicians

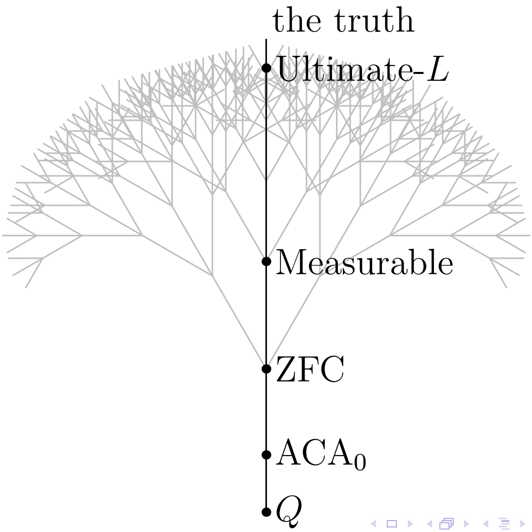
- Most schools of philosophy of mathematics fight to defend the legitimateness of everything that mathematicians might interest.
- Even the contemporary constructivists place most of their efforts on showing that many classical results can be done in their restrictive systems.

The philosophical ideas we do concern

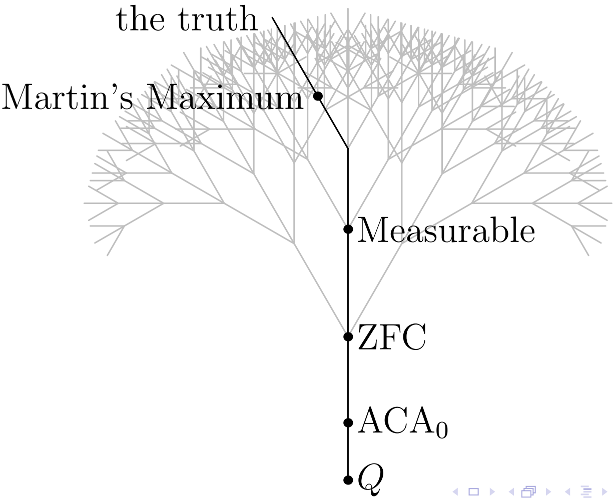
To investigate the real impacts of the philosophy of mathematics on the mathematical research, we are particularly interested in the following ideologies:

- Realism (Gödel, Californian School)
- Formalism (Israeli School)
- Constructivism (Nelson, Havey Friedman, etc.)

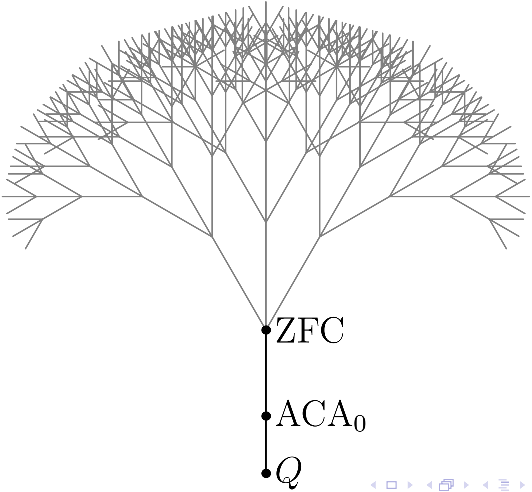
Realism



Realism, another version



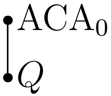
Formalism (based on ZFC)



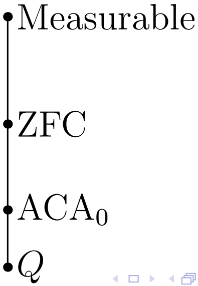
Constructivism: Strict Finitism

$\bullet Q$

Constructivism: Predicativism



Constructivism: Friedman's



Axiom of Choice

Axiom of Choice (AC): Every set can be well-ordered.

Banach-Tarski paradox:

Key: Those pieces are Lebesgue nonmeasurable sets, whose existence follows from AC.

The properties of regularity

Mathematicians choose to tolerant AC and believe that simple and nature sets will not behave wildly, they have the properties of regularity.

- Complexity of sets of reals: Borel hierarchy, Projective hierarchy

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Projective hierarchy

$$
\begin{array}{ccccccccccc}
 & & \mathbf{\Sigma}^1_1 & & & & \mathbf{\Sigma}^1_2 & \cdots & & & \mathbf{\Sigma}^1_n & & & \cdots \\
 & \nearrow & & \searrow & & \nearrow & & & & \nearrow & & \searrow & & \\
\mathbf{\Delta}^1_1 & & & & \mathbf{\Delta}^1_2 & & & \cdots & \mathbf{\Delta}^1_n & & & & \mathbf{\Delta}^1_{n+1} & \cdots \\
 & \searrow & & \nearrow & & \searrow & & & & \searrow & & \nearrow & & \\
 & & \mathbf{\Pi}^1_1 & & & & \mathbf{\Pi}^1_2 & \cdots & & & \mathbf{\Pi}^1_n & & & \cdots
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The properties of regularity

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- Complexity of sets of reals: Borel hierarchy, Projective hierarchy
- Properties of regularity:
  - Lebesgue measurable
  - Property of Baire
  - Perfect set property

Infinite Games on ω with perfect information

Two-person games on ω of length ω with perfect information:

player I:	a_0		a_2		$\ldots$	
player II:		a_1		a_3		$\ldots$

Given $A \subseteq \omega^\omega$, player I win the game G_A if the play $\langle a_i : i \in \omega \rangle \in A$.

Axioms of Determinacy

Let $A \subseteq \omega^\omega$

- Set $A \subseteq \omega^\omega$ is determined if the either player I or player II has a winning strategy for the ω game on A.
- AD: every set of reals is determined.
- PD: every projective set is determined.

Axioms of Determinacy

- AD proves all properties of regularity.
- Thus AD is inconsistent with ZFC.
- Determinacy proves the properties of regularity locally, e.g. Σ^1_n-D implies every Σ^1_{n+1} set is Lebesgue measurable.
- Thus PD secures all properties of regularity for all projective sets.
- Is PD consistent, or can we even prove it?

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Unprovability of PD

Theorem

Assuming $V = L$, then PD is false. Thus

$$\text{ZFC} \vdash \text{Con(ZFC)} \rightarrow \text{Con(ZFC} + \neg\text{PD)}$$

A formalist may not be interested in such "unprovable" statements as PD any more.

However, realists would still try to justify PD by proving it from some plausible extension of ZFC.

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The proof of PD

- Solovay 1964: The consistency strength of PD is beyond measurable.
- Martin 1969: Measurable implies $\mathbf{\Pi}^1_1$-D.
- Martin 1978: ω-huge implies $\mathbf{\Pi}^1_2$-D.
- Woodin 1984: I0 implies PD
- Woodin 1988: Supercompact implies PM
- Martin-Steel 1988: infinity many Woodin implies PD

The proof of PD, a review

Since Woodin cardinals are well justified large cardinal axioms, and PD resolves nearly all independent problem concerning $P(\mathbb{N})$ in a highly plausible way, many set theorists recognize PD as a missing truth.

Note:

- Only realists regard a proof from LCAs as a justification.
- Proofs from very strong LCAs helped people to find the best result.

Practically, any kind of constructivism may obstacle the discovery

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Conclusion

Philosophy of mathematics may have impacts on the research of mathematics at least in the following sense:

- Gives motivation
- Makes reasonable conjectures
- Sets obstacles

Thank you!