Strong Axioms of Infinity and the Continuum Function

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Introduction

What are *strong axioms of infinity*, or *large cardinal axioms*?

- There are many degrees of infinity in mathematics:

$$|\mathbb{N}| = \omega = \aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots \aleph_{\aleph_1} < \cdots \aleph_{\aleph_\omega} \cdots$$

- Strong axioms of infinity are set theoretic axioms asserting that there is a dividing line like the one between the finite and the infinite, but higher up.
- For example, the first infinite cardinal ω has the following properties.
  1. For every $n < \omega$ we have $|P(n)| < \omega$.
  2. Every finite set $X \subseteq \omega$ is bounded.

- We say that a cardinal $\kappa > \omega$ is *inaccessible* if the following hold.
  1. For every $\alpha < \kappa$ we have $|P(\alpha)| < \kappa$.
  2. Every set $X \subseteq \kappa$ with $|X| < \kappa$ is bounded in κ.
- The assertion "there exists an inaccessible cardinal" is an example of a large cardinal axiom, or strong axiom of infinity.

$$\text{finite} < \omega \leq \text{infinite}$$

$$\text{things smaller than } \kappa < \kappa \leq \text{things of size at least } \kappa$$

- The theory ZFC + "there is an inaccessible cardinal" can prove $\mathrm{CON}(\mathrm{ZFC})$ as well as $\mathrm{CON}(\mathrm{ZFC} + \mathrm{CON}(\mathrm{ZFC}))$, etc.
- The theory ZFC + "there is a Mahlo cardinal" can prove $\mathrm{CON}(\mathrm{ZFC} + \text{"there is an inaccessible cardinal"})$.

0=1
I0—I3
n-huge
huge
almost huge
extendible
supercompact
strongly compact
strong
$o(\kappa) = \kappa^{++}$
measurable
Ramsey
Rowbottom
indescribable
weakly compact
Mahlo
inaccessible

0=1
I0—I3
n-huge
huge
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supercompact
strongly compact
strong
$o(\kappa) = \kappa^{++} \longleftrightarrow$ "singular stong limit γ with $2^\gamma > \gamma^+$"
measurable
Ramsey
Rowbottom
indescribable
weakly compact
Mahlo
inaccessible $\longleftrightarrow$ "every set of reals is Lebesgue measurable"

Set theorists want to understand the structure of large cardinals, their combinatorial properties, and how they are related to one another.

One way to investigate these properties is by using forcing to try and change things about the universe (like CH) while preserving large cardinals.

Recall:

- CH says $2^\omega = \aleph_1$. Equivalently, there are no intermediate degrees of infinity between $\omega = |\mathbb{N}|$ and $2^\omega = |P(\mathbb{N}) = |\mathbb{R}|$.
- GCH says for all cardinals κ we have $2^\kappa = \kappa^+$.

Outline

The idea

Idea of forcing:

- Given a model of set theory $M \models \mathrm{ZFC}$, a partial order $(\mathbb{P}, \leq) \in M$, and a special filter $G \subseteq \mathbb{P}$, there is a model $M[G] \models \mathrm{ZFC}$ called the *forcing extension of* M *by* $\mathbb{P}$, where $M \subsetneq M[G]$.
- The partial order $\mathbb{P}$, is chosen in such a way that $M[G]$ has some desired property.
- We always assume that partial orders $(\mathbb{P}, \leq)$ have a greatest element denoted by $\mathbb{1}$.
- Elements $p \in \mathbb{P}$ are called *conditions* because they are thought of as holding some partial amount of information about a filter G if $p \in G$.

Definition

Let $(\mathbb{P}, \leq)$ be a partial order in a model $M \models \mathrm{ZFC}$.

- We say that $D \subseteq \mathbb{P}$ is **dense** if for every condition $p \in \mathbb{P}$ there is a stronger condition $q \leq p$ with $q \in D$.
- We say that $G \subseteq \mathbb{P}$ is a **filter** if the following hold.
  1. Whenever $p, q \in G$ there is an $r \in G$ such that $r \leq p$ and $r \leq q$.
  2. If $p \in G$ and $q \in \mathbb{P}$ with $p \leq q$ then $q \in G$.
- We say that $G \subseteq \mathbb{P}$ is an M-**generic filter** if it intersects every dense subset D of $\mathbb{P}$ with $D \in M$.

Fact

If M is a countable model of ZFC and $(\mathbb{P}, \leq) \in M$ is a partial order, then there is an M-generic filter for $\mathbb{P}$.

Proof.

Let $D_0, D_1, D_2, \ldots$ be an enumeration of the countably many dense subsets of $\mathbb{P}$ that are in M. Let $p \in \mathbb{P}$ be any condition.

- Since D_0 is dense there is a $p_0 \leq p$ with $p_0 \in D_0$.
- D_1 dense $\implies$ there is a $p_1 \leq p_0$ with $p_1 \in D_1$.
- Continuing by induction we can find a sequence

  $$\cdots p_3 \leq p_2 \leq p_1 \leq p_0$$

  such that $p_i \in D_i$.
- Let $G = \{r \in \mathbb{P} \mid \exists n(p_n \leq r)\}$.

□

Example: (Forcing to add a subset to ω.)

- Let M be a countable (transitive) model of ZFC.
- Consider the partial order $(\mathbb{P}, \leq)$ where

$$\mathbb{P} = \{f \mid f \text{ is a finite partial function from } \omega \text{ to } 2\}$$

  is ordered by $f \leq g \iff f \supseteq g$.
- Suppose $G \subseteq \mathbb{P}$ is an M-generic filter, meaning:
  - If $f, g \in G$ then f and g are compatible as functions.
  - If $f \in G$ and $f \leq g$ then $g \in G$.
  - G intersects every dense subset of $\mathbb{P}$ in M.
- Then $f_G = \bigcup G = \bigcup_{f \in G} f$ is a function with $dom(f_G) \subseteq \omega$.

Fact

$\mathrm{dom}(f_G) = \omega$ *and so* $f_G : \omega \to 2$.

Proof.

Fix $n \in \omega$ and let $D_n = \{f \in \mathbb{P} \mid n \in \mathrm{dom}(f)\} \in M$. The set $D_n \subseteq \mathbb{P}$ is dense because if $p \in \mathbb{P}$ then we can always extend p to a larger function $f \supseteq p$ (if necessary) such that $n \in \mathrm{dom}(f)$, and then $f \in D_n$. Since D_n is dense we have $G \cap D_n \neq \emptyset$ because G is M-generic. This means that there is some finite partial function $h \in G \cap D_n$ with $n \in \mathrm{dom}(h)$ and thus $n \in \mathrm{dom}(f_G) = \bigcup_{f \in G} f$. □

Fact

$G \notin M$

Proof.

For a contradiction suppose that $G \in M$. Then $\mathbb{P} \setminus G \in M$ is dense because if $p \in \mathbb{P}$ then we can find a stronger condition $q \leq p$ that is not compatible with some element of G, and since G is a filter this implies that $q \notin G$ and thus $q \in \mathbb{P} \setminus G$. □

- Since $G \notin M$ it follows that $f_G \notin M$.
- $f_G : \omega \to 2$.
- The "*Forcing Theorems*" gives us a model $M[G] \supsetneq M$ of ZFC with $G \in M[G]$.
- Then $f_G \in M[G]$ is a **new** subset of ω because $f_G \notin M$.
- It follows that $P(\omega)^M \subsetneq P(\omega)^{M[G]}$.
- Terminology: Suppose $f = \{(0,7),(1,12),(2,5)\} \in \mathbb{P}$. If $f \in G$ then $f \subseteq f_G$ and in particular $f_G(1) = 12$. In this case we say that "the condition f forces $f_G(1) = 12$."

Notation: The forcing partial order we just used is written as $\mathbb{P} = \mathrm{Add}(\omega, 1)$. $\mathrm{Add}(\omega, 1)$ is the forcing for adding one Choen subset to ω.

Example: (Forcing to add many subsets to ω.)

- Let us define a partial order $(\mathbb{P}, \leq)$ working in some model $M \models \mathrm{ZFC} + \mathrm{CH}$.

  $$\mathbb{P} = \{p \mid p \text{ is a finite partial function from } \omega_2 \times \omega \text{ to } \{0,1\}\}$$

  ordered by $p \leq q \iff p \supseteq q$.
- Suppose G is M-generic for $\mathbb{P}$ and let $F = \bigcup_{p \in G} p$.

Fact

$\mathrm{dom}(F) = \omega_2 \times \omega$ *so that* F *is a function* $\omega_2 \times \omega \to \{0,1\}$.

Proof.

This is essentially the same as before. Fix $(\alpha, n) \in \omega_2 \times \omega$ and let $D_{(\alpha,n)} = \{p \in \mathbb{P} \mid (\alpha, n) \in p\}$. It follows that $D_{(\alpha,n)}$ is dense and thus $G \cap D_{(\alpha,n)} \neq \emptyset$. This means that some $p \in G$ has $(\alpha, n) \in \mathrm{dom}(p)$. Therefore $(\alpha, n) \in \mathrm{dom}(F)$. □

- $F \notin M$ (just like before $G \notin M$).
- The "*Forcing Theorems*" gives us a model $M[G] \supsetneq M$ with $F \in M[G]$.
- For each $\alpha < \omega_2$ let F_α be the α-th slice of F:

$$F_\alpha(n) = i \text{ if and only if } F(\alpha, n) = i.$$

- For each $\alpha < \omega_2$ we have $F_\alpha \notin M$ but $F_\alpha \in M[G]$.

Fact

If $\alpha, \beta < \omega_2$ and $\alpha \neq \beta$ then $F_\alpha \neq F_\beta$.

Proof.

Fix $\alpha < \beta < \omega_2$. Then the set $D_{\alpha,\beta} = \{p \in \mathbb{P} \mid \exists n(F_\alpha(n) \neq F_\beta(n)\}$ is dense. So there is some condition $p \in G$ and number $n < \omega$ such that $p(\alpha, n) \neq p(\beta, n)$. This implies that $F_\alpha(n) \neq F_\beta(n)$ since $F = \bigcup_{p \in G} p$. □

- So in $M[G]$ there is a collection $\{F_\alpha \mid \alpha < \omega_2\}$ such that each F_α yields a distinct subset of ω.
- In $M[G]$ do we now have $|P(\omega)| \geq \omega_2$ and therefore $\neg\mathrm{CH}$?
- Not quite: what we have shown so far is that $M[G] \models |P(\omega)| \geq (\omega_2)^M$ but in order to have $M[G] \models \neg\mathrm{CH}$ we would need $M[G] \models |P(\omega)| \geq (\omega_2)^{M[G]}$.

Remark

We can show that $(\omega_2)^M = (\omega_2)^{M[G]}$ by using the fact $\mathbb{P}$ has the property that every collection $A \subseteq \mathbb{P}$ of pairwise incompatible elements must be countable.

Definition

- Conditions $p, q \in \mathbb{P}$ are called *incompatible* if they have no common extension: $\neg\exists r \in \mathbb{P}(r \leq p, q)$.
- We say that $A \subseteq \mathbb{P}$ is an *antichain* if for all $p, q \in \mathbb{P}$, p, q are incompatible:

$$(\forall p, q \in A)(\neg\exists r \in \mathbb{P})(r \leq p, q).$$

- A partial order $\mathbb{P}$ is called *c.c.c. (countable chain condition)* if whenever $A \subseteq \mathbb{P}$ is an antichain we have $|A| \leq \omega$.

Fact

The poset $\mathbb{P} = \mathrm{Add}(\omega, \omega_2)$ satisfies c.c.c. (by a Δ-system argument).

Fact

If $\mathbb{P}$ is c.c.c. then $\mathbb{P}$ preserves cardinals (hence $(\omega_2)^M = (\omega_2)^{M[G]}$).

Back to the **example:**

- In $M[G]$ we have a collection $\{F_\alpha \mid \alpha < \omega_2\}$ of ω_2 many *distinct* and *new* subsets of ω.
- Since $\mathbb{P}$ is c.c.c. we have $(\omega_2)^M = (\omega_2)^{M[G]}$ and it follows that

$$M[G] \models 2^\omega = \omega_2$$

or

$$M[G] \models \neg \mathrm{CH}.$$

Notation: The forcing we did in this example is written as $\mathbb{P} = \mathrm{Add}(\omega, \omega_2)$. It is the forcing for adding ω_2 Cohen subsets to ω.

What is $M[G]$?

The previous examples describe the idea behind forcing, but how do we define the model $M[G] \models \mathrm{ZFC}$?

- Roughly, $M[G]$ is the collection of all sets that can be constructed from G by set theoretic operations, like union, intersection, complements, etc.
- Precisely, we define $M[G]$ using "names."
- τ is called a $\mathbb{P}$-*name* if τ is a collection of ordered pairs such that

$$\forall(\sigma, p) \in \tau(\sigma \text{ is a } \mathbb{P}\text{-name and } p \in \mathbb{P})$$

- Given a $\mathbb{P}$-name τ and a generic filter $G \subseteq \mathbb{P}$, we define the evaluation of τ by G as

$$\tau_G = \{\sigma_G \mid (\exists p \in G)\ (\sigma, p) \in \tau\}$$

Fact

Every set $x \in M$ has a $\mathbb{P}$ name $\check{x}$ such that $\check{x}_G = x$.

Proof.

- Fix $x \in M$ and by induction assume the fact is true for all $y \in x$.
- The following is a $\mathbb{P}$-name.

$$\check{x} = \{(\check{y}, \mathbb{1}) \mid y \in x\}$$

(For each $y \in x$, put $\check{y}$ into $\check{x}$ with probability $\mathbb{1}$.)

- Furthermore, $\check{x}_G = x$ because

$$\begin{aligned} y \in x &\iff (\check{y}, \mathbb{1}) \in \check{x} \\ &\iff y = \check{y}_G \in \check{x}_G \qquad \text{(since } \mathbb{1} \in G\text{)} \end{aligned}$$

□

Definition

Suppose $\mathbb{P} \in M$ is a partial order and $G \subseteq \mathbb{P}$ is M-generic. Define

$$M[G] = \{\tau_G \mid \tau \text{ is a } \mathbb{P}\text{-name}\}$$

Theorem (Forcing Theorem 1)

Let M be a (countable) transitive model of ZFC and let $(\mathbb{P}, \leq) \in M$ be a partial order. If $G \subseteq \mathbb{P}$ is M-generic for $\mathbb{P}$ then there is a transitive model $M[G]$ such that:

(i) $M[G] \models \mathrm{ZFC}$,

(ii) $M \subseteq M[G]$ *and* $G \in M[G]$, *and*

(iii) $\mathrm{ORD}^{M[G]} = \mathrm{ORD}^M$.

- $M \subseteq M[G]$ is easy because if $x \in M$ then $x = \check{x}_G \in M[G]$.
- To show that $M[G]$ satisfies the pairing axiom,

"if x and y are sets then $\{x, y\}$ is a set"

suppose $\tau_G, \sigma_G \in M[G]$ and let $\rho = \{(\tau, \mathbb{1}), (\sigma, \mathbb{1})\}$. Then ρ is a $\mathbb{P}$-name and we have $\rho_G = \{\tau_G, \sigma_G\}$ so that the pair $\{\tau_G, \sigma_G\} \in M[G]$.

Definition

Suppose $(\mathbb{P}, \leq)$ is a partial order in some model M. We say that a condition $p \in \mathbb{P}$ *forces* a sentence φ in the "forcing language" if and only if whenever $G \subseteq \mathbb{P}$ is M-generic an $p \in G$ we have $V[G] \models \varphi$. We write $p \Vdash \varphi$ to mean "p forces φ."

- A sentence in the forcing language:

$$\tau \subseteq \check{\omega} \times \check{\omega} \wedge \forall x \in \check{\omega} \exists! y \in \check{\omega}((x, y) \in \tau)$$

Theorem (Forcing Theorem 2)

"Everything true statement in a forcing extension $M[G]$ is forced to be true."

$$M[G] \models \varphi \iff \exists p \in G(p \Vdash \varphi)$$

Easton's theorem

- We can force $2^\omega = \omega_7$ using the same method as above, just replace each "ω_2" with "ω_7" and the argument works.
- Can we force $2^\omega = \omega_7$ and $2^{\omega_3} = \omega_{10}$?
- Can we force $2^\omega = \omega_7$ and $2^{\omega_3} = \omega_6$?
- In general, what properties does ZFC impose on the behavior of the continuum function $\kappa \mapsto 2^\kappa$?

Theorem (König)

If $\kappa < \mathrm{cf}(2^\kappa) \leq 2^\kappa$.

Fact

(Monotonicity) If $\kappa < \lambda$ *are cardinals then* $2^\kappa \leq 2^\lambda$.

Shortly after Cohen proved $\neg\mathrm{CH}$ to be consistent, Easton proved that the above two restrictions on the continuum function are *the only restrictions* on the regular cardinals under ZFC.

Theorem (Easton)

Suppose GCH *holds and* F *is a function from the regular cardinals to the cardinals satisfying the following.*

$$\kappa < \mathrm{cf}(F(\kappa)) \tag{E1}$$

$$\kappa < \lambda \implies F(\kappa) \leq F(\lambda) \tag{E2}$$

Then there is a cardinal preserving forcing extension $V[G]$ *in which* $2^\kappa = F(\kappa)$ *for all regular cardinals* κ*.*

Preserving cardinals in Easton's theorem:

Definition

Let $\mathbb{P}$ be a partial order.

- $\mathbb{P}$ is κ^+-*c.c.* if every antichain $A \subseteq \mathbb{P}$ has at most κ, $|A| \leq \kappa$.
- $\mathbb{P}$ is $<\kappa$-*closed* if every decreasing sequence of conditions $\langle p_\alpha \mid \alpha < \gamma \rangle$ of length $\gamma < \kappa$ has a lower bound:

$$p \leq \cdots \leq p_{\alpha+1} \leq p_\alpha \leq \cdots p_1 \leq p_0.$$

Fact

Assuming GCH, *if* $\mathbb{P}$ *is* κ^+-*c.c. and* $<\kappa$-*closed then* $\mathbb{P}$ *preserves cardinals.*

Outline

The idea

An *elementary embedding* $j : \mathcal{M} \to \mathcal{N}$ from one L-structure to another is a function $j : M \to N$ which respects "truth" in the sense that for any $a \in M$ and any L-formula with one free variable we have

$$\mathcal{M} \models \varphi(a) \iff \mathcal{N} \models \varphi(j(a)).$$

- The natural embedding $f : (\mathbb{Z}, 0, 1, +, \cdot, \leq) \to (\mathbb{Q}, 0, 1, +, \cdot, \leq)$ respects the truth of quantifier free sentences but does not preserve the truth of sentences containing quantifiers.
- Ultrapowers provide natural examples of elementary embeddings between first order structures.

Suppose $\{\mathcal{M}_i \mid i \in I\}$ is a collection of L-structures and U is an ultrafilter on I.

Let $M_I = \prod_{i \in I} M_i$. So if $f \in M_I$ then f can be viewed as a function with domain I such that for each $i \in I$ we have $f(i) \in M_i$.

Define an equivalence relation

$$f \sim_U g \iff \{i \in I \mid f(i) = g(i)\} \in U.$$

For each function $f \in M_I$, let $[f]_U = \{g \in M_I \mid f \sim_U g\}$ denote the equivalence class of f.

Given a collection of L-structures $\{\mathcal{M}_i \mid i \in I\}$ and an ultrafilter U on I, the ultraproduct of the $\mathcal{M}_i$ with respect to U is an L-structure $\mathcal{M}_I/U$ with domain

$$\prod_{i \in I} \mathcal{M}_i/U = \{[f]_U \mid \mathrm{dom}(f) = I, f(i) \in M_i\}.$$

For example, if f is a function symbol in L then

$$f^{\mathcal{M}_I/U}([\langle g_i \mid i \in I\rangle]_U) = [\langle f^{\mathcal{M}_i}(g_i) \mid i \in I\rangle]_U.$$

Theorem (Łoś' Theorem)

Suppose that I is a nonempty set, $\{\mathcal{M}_i \mid i \in I\}$ is a collection of L-structures, $M_I = \prod_{i \in I} M_i$, and U is an ultrafilter on I. Then, for every L-formula φ with one free variable and every $a \in M_I$ we have

$$\mathcal{M}_I/U \models \varphi([a]_U) \iff \{i \in I \mid \mathcal{M}_i \models \varphi(a(i))\} \in U.$$

Łoś' Theorem gives us an elementary embedding from a structure into its ultrapower with respect to some ultrafilter.

- Suppose $\mathcal{M}$ is a first order structure and U is an ultrafilter on some index set I.
- Let $\mathcal{M}^I/U$ be the corresponding **ultrapower**. For each $i \in I$, $\mathcal{M}_i = \mathcal{M}$.

$$M^I/U = \{[f]_U \mid f : I \to M\}$$

- For each $a \in M$ let $c_a : I \to M$ be the constant function defined by $c_a(i) = a$ for every $i \in I$.
- Then the function $j : M \to M^I/U$ defined by $j(a) = [c_a]_U$ is an elementary embedding.
- If φ is a first order formula with one free variable and $a \in M$ then we have

$$\begin{aligned} \mathcal{M} \models \varphi(a) &\iff \{i \in I \mid \mathcal{M}_i = \mathcal{M} \models \varphi(c_a(i))\} = I \\ &\iff \mathcal{M}^I/U \models \varphi([c_a]_I) \qquad \text{(by Łoś' Theorem)} \\ &\iff \mathcal{M}^I/U \models \varphi(j(a)) \end{aligned}$$

Measurable cardinals

Definition

A cardinal $\kappa > \omega$ is called a *measurable cardinal* if there is an ultrafilter U on κ satisfying the following properties.

- (nonprinciple) No subset of κ generates U, in other words, for every $A \subseteq \kappa$ we have $U \neq \{X \subseteq \kappa \mid A \subseteq X\}$.
- (κ-complete) U is closed under intersections of length less than κ, in other words, if $\{X_\alpha \mid \alpha < \gamma\}$ is a collection of subsets of κ with $X_\alpha \in U$ and $\gamma < \kappa$ then $\bigcap_{\alpha<\gamma} X_\alpha \in U$.

If U is a nonprinciple κ-complete ultrafilter on κ we call U a *measure* on κ.

Working in a model of set theory $V \models \mathrm{ZFC}$, if κ is a measurable cardinal and U is a measure on κ, then we can form the ultrapower of the entire universe V^κ/U as follows.

- For two functions $f, g : \kappa \to V$ with $f, g \in V$ define

$$f \sim_U g \iff \{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in U.$$

- Define $[f]_U = \{g \in V^\kappa \mid g \sim_U f\}$.
- Also, define $\in^{V^\kappa/U}$ by

$$[f]_U \in [g]_U \iff \{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\} \in U.$$

- Elements of the ultrapower V^κ/U are equivalence classes of functions from κ to V.
- Notice that such equivalence classes could be proper classes, but this problem can be avoided using Scott's trick of letting $[f]_U$ consist of only those functions g equivalent to f of *minimal Levy rank*.

This ultrapower provides an elementary embedding $j : V \to V^\kappa/U$ defined by $j(x) = [c_x]_U$.

V^κ/U is *well-founded* meaning that there is no infinite decreasing $\in$-chain

$$\cdots \in [f_{n+1}]_U \in [f_n]_U \in \cdots \in [f_1]_U \in [f_0]_U.$$

If there were such a decreasing $\in$-chain then for every $n < \omega$ we have $[f_{n+1}]_U \in [f_n]_U \implies X_n = \{\alpha < \kappa \mid f_{n+1}(\alpha) \in f_n(\alpha)\} \in U$. But then by the κ-completeness of U it follows that $X_\omega = \bigcap_{n<\omega} X_n \in U$. In particular there is some $\alpha \in X_\omega$ and then we would have a decreasing $\in$-chain in V

$$\cdots \in f_{n+1}(\alpha) \in f_n(\alpha) \in \cdots \in f_1(\alpha) \in f_0(\alpha)$$

which contradicts the fact that $V \models \mathrm{ZFC}$.

Because the ultrapower V^κ/U is well-founded we can take the transitive collapse (Mostowski collapse) to get a transitive class M with $V^\kappa/U \cong M$.

Typically the ultrapower V^κ/U is identified with this transitive class M and we write the elementary embedding as $j : V \to M$. Since M is transitive it follows that V and M have the same ordinals.

Question: what does j do to the ordinals?

- For any ordinal α we have $\alpha \leq j(\alpha)$ because if for some ordinal α we had $j(\alpha) < \alpha$, then assume α is the least such ordinal and let $\delta = j(\alpha)$. Then by assumption $\delta < \alpha$ and by elementarity we have $j(\delta) < j(\alpha)$, a contradiction.
- If $\alpha < \kappa$ then $j(\alpha) = \alpha$.

- $j(\kappa) > \kappa$. We know that $j(\kappa) \geq \kappa$ so we just have to show that $j(\kappa) \neq \kappa$. Let $\mathrm{id} : \kappa \to \kappa$ be the identity function defined by $\mathrm{id}(\alpha) = \alpha$ for each $\alpha < \kappa$. It follows that $[\mathrm{id}]_U \in j(\kappa)$ and $[\mathrm{id}]_U \notin \kappa$.

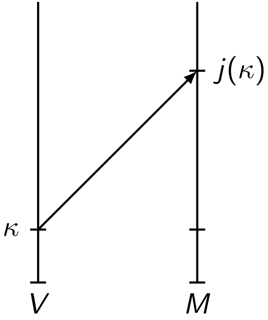


Figure : The placement of relevant ordinals.

Definition

κ is called the *critical point* of an elementary embedding $j : V \to M$ if it is the least ordinal moved by j.

Fact

κ is a measurable cardinal if and only if there is an elementary embedding $j : V \to M$ with critical point κ.

Proof.

We already proved ($\Longrightarrow$). To prove ($\Longleftarrow$) suppose $j : V \to M$ is an elementary embedding with critical point κ. Let $\boxed{U = \{X \subseteq \kappa \mid \kappa \in j(X)\}}$ and check that U is a κ-complete nonprinciple ultrafilter on κ. □

The measure U obtained from an elementary embedding j as in the above proof has the additional property that when we take the ultrapower by U to get $j_U : V \to M_U$ then $[\mathrm{id}]_U = \kappa$ and every element $x \in M$ is of the form $x = [f]_U = j_U(f)(\kappa)$ for some function $f : \kappa \to V$.

$$M_U = \{j_U(f)(\kappa) \mid f : \kappa \to V, f \in V\}$$

Such a measure U is called a *normal* measure on κ.

Fact

If $j : V \to M$ is the ultrapower by a normal measure on κ then $M^\kappa \subseteq M$ in V.

Fact

If κ is a measurable cardinal and GCH fails at κ: $2^\kappa = \kappa^{++}$, then GCH must fail at many cardinals less than κ.

Proof.

Let $j : V \to M$ be an ultrapower by a normal measure on κ with $M^\kappa \subseteq M$. The measure satisfies $U = \{X \subseteq \kappa \mid \kappa \in j(X)\}$. Then

$$\kappa^+ = (\kappa^+)^M < \kappa^{++} = 2^\kappa \leq (2^\kappa)^M.$$

This means that $M \models$ "GCH fails at κ." Therefore,

$$M \models \kappa \in j(\{\alpha < \kappa \mid 2^\alpha > \alpha^+\})$$

and this implies that $\{\alpha < \kappa \mid 2^\alpha > \alpha^+\} \in U$. □

Example. (Ghost coordinates) Suppose $j : V \to M$ is an elementary embedding with critical point κ. So, in particular, κ is a measurable cardinal.

- Fix some $\gamma < \kappa = \mathrm{crit}(j)$. Then $j(\gamma) = \gamma$.
- Suppose $\vec{x} = \langle x_\alpha \mid \alpha < \gamma \rangle \in V$ is a sequence of length γ.
- Then, by elementarily, $j(\vec{x})$ is a sequence of length $j(\gamma)$, but $j(\gamma) = \gamma$. So $j(\vec{x}) = \langle y_\alpha \mid \alpha < \gamma \rangle$.
- For $\alpha < \gamma$, the α-th element of $\vec{x}$ is x_α. So, by elementarity, the $j(\alpha) = \alpha$-th element of $j(\vec{x})$ is $j(x_\alpha)$. In other words, $y_\alpha = j(x_\alpha)$.
- Now, if $\vec{z} = \langle z_\alpha \mid \alpha < \kappa \rangle$ is a sequence of length κ, then $j(\vec{z}) = \langle w_\alpha \mid \alpha < j(\kappa) \rangle$ and for $\alpha < \kappa$ we have $w_\alpha = j(z_\alpha)$.
- But, the sequence $j(\vec{z})$ has "new" coordinates, so called "ghost coordinates" for $\kappa \leq \alpha < j(\kappa)$.

Supercompactness

Supercompactness is a natural generalization of measurability.

Definition

Suppose $\kappa \leq \lambda$.

- A cardinal κ is called λ-*supercompact* if there is an elementary embedding $j : V \to M$ with critical point κ such that $M^\lambda \subseteq M$ and $j(\kappa) > \lambda$.
- κ is called *supercompact* if it is λ-supercompact for every cardinal $\lambda > \kappa$.

The λ-supercompactness of a cardinal κ can be characterized by the existence of a certain type of ultrafilter on the set

$$P_\kappa \lambda := \{x \subseteq \lambda \mid |x| < \kappa\}.$$

If $j : V \to M$ witnesses that κ is λ-supercompact then

$$U = \{X \subseteq P_\kappa\lambda \mid j``\lambda \in j(X)\}$$

is a "κ-complete nonprinciple normal fine ultrafilter" on the set $P_\kappa\lambda$.

If $j_U : V \to M_U$ is the ultrapower by such an ultrafilter on $P_\kappa\lambda$ then $M_U^\lambda \subseteq M_U$, $j_U(\kappa) > \lambda$, and

$$M_U = \{j(f)(j``\lambda) \mid f : P_\kappa\lambda \to V, f \in V\}.$$

Tallness and extenders

Definition

κ is *λ-supercompact with tallness θ* if there is an elementary embedding $j : V \to M$ with critical point κ such that $M^\lambda \subseteq M$ and $j(\kappa) > \theta$.

From such an embedding we can define a system of ultrafilters. If $j : V \to M$ is an embedding obtained from these ultrafilters then

$$M = \{j(f)(j\text{''}\lambda, \alpha) \mid f : P_\kappa\lambda \times \kappa \to V, f \in V, \alpha < \theta\}.$$

Outline

The idea

Idea: use forcing to fine tune the universe while preserving large cardinals.

Example: Not only is CH independent from ZFC, but CH is independent from ZFC + "strong axioms of infinity."

Theorem (Levy-Solovay)

Suppose κ is a measurable cardinal and $\mathbb{P}$ is a partial order with $|\mathbb{P}| < \kappa$. If $G \subseteq \mathbb{P}$ is V-generic then κ remains measurable in $V[G]$.

Proof.
Recall: κ is measurable if and only if there is an elementary embedding $j : V \to M$ with critical point κ.
Key Idea: To argue that κ remains measurable in $V[G]$, we lift an elementary embedding $j : V \to M$ to the forcing extension $j^* : V[G] \to M[j(G)]$.

We have an embedding $j : V \to M$. How should we define the embedding $j^* : V[G] \to M[j(G)]$?

Every $x \in V[G]$ is of the form $x = \tau_G$ for some $\mathbb{P}$-name $\tau \in V$. Since $\tau \in V$ is a $\mathbb{P}$-name, by elementarily $j(\tau)$ is a $j(\mathbb{P})$ name. If we had a generic filter K for $j(\mathbb{P})$ then we could evaluate $j(\tau)$ using K.

Since $|\mathbb{P}| < \kappa$ and κ is the critical point of j, it follows that $j(\mathbb{P}) = \{j(p) \mid p \in \mathbb{P}\}$. Let K be the filter generated by $j\text{''}G = \{j(p) \mid p \in G\}$.

Let's show that K is M-generic for $j(\mathbb{P})$. If $D \in M$ is a dense subset of $j(\mathbb{P})$ then it follows that $D = \{j(p) \mid p \in D_0\}$ where D_0 is a dense subset of $\mathbb{P}$. Since G is V-generic there is a condition $p_0 \in G \cap D_0$. Then $j(p_0) \in j\text{''}G \cap D \subseteq K \cap D$.

Now define $j^*(x) = j(\tau_G) = j(\tau)_K$.

Let us check that j^* is well-defined. Suppose $\dot{x}$ and $\dot{y}$ are $\mathbb{P}$-names and that $\dot{x}_G = \dot{y}_G$. We need to show that $j(\dot{x})_K = j(\dot{y})_K$. Choose $q \in G$ such that $q \Vdash \dot{x} = \dot{y}$. Applying j and using elementarily we conclude that $j(q) \Vdash j(\dot{x}) = j(\dot{y})$. Since $\boxed{j\text{''}G \subseteq K}$ it follows that $j(q) \in K$ and thus $j(\dot{x})_K = j(\dot{y})_K$.

Exercise: Check that j^* is an elementary embedding (it's similar). This completes the proof. $\square$

Notice: This shows that CH is independent from ZFC + "there is a measurable cardinal."

Remark: Many times, one shows that a certain forcing preserves a large cardinal axiom by "lifting an embedding" from $j : V \to M$ to $j^* : V[G] \to M[j(G)]$.

Lifting large cardinal embeddings

Lemma (The Lifting Criterion)

Suppose $j : M \to N$ is an elementary embedding and $\mathbb{P} \in M$ is a poset. Let $G \subseteq \mathbb{P}$ be M-generic and $K \subseteq j(\mathbb{P})$ be N-generic. Then j lifts to $j^* : M[G] \to N[K]$ if and only if $\boxed{j\text{"} G \subseteq K}$.

Proof.

Just define $j^*(\tau_G) = j(\tau)_K$ as before. □

Next: Is it consistent that GCH fails at a measurable cardinal? In other words, is the theory ZFC + "there is a measurable cardinal κ such that $2^\kappa = \kappa^{++}$" consistent?

- This uses the idea of "master condition."

Theorem (Silver)

Suppose κ is κ^{++}-supercompact and GCH holds. Then there is a forcing extension in which κ is measurable and $2^\kappa = \kappa^{++}$.

- The forcing is of the form $\mathbb{P} * \mathbb{Q} = \mathbb{P} * \mathrm{Add}(\kappa, \kappa^{++})$.
- Let $j : V \to M$ witness that κ is κ^{++}-supercompact. So, $\mathrm{crit}(j) = \kappa$, $M^{\kappa^{++}} \subseteq M$, and $j(\kappa) > \kappa^{++}$.
- Suppose $G * H$ is V-generic for $\mathbb{P} * \mathbb{Q}$.
- We can lift j to $j : V[G] \to M[j(G)]$ where $j(G)$ satisfies $j\text{``}G \subseteq j(G)$.

- Now we need to lift j further through $\mathbb{Q}$. So we need an $M[j(G)]$-generic filter $K \subseteq j(\mathbb{Q})$ such that $j"H \subseteq K$.
- Consider the forcing $\mathbb{Q} = \mathrm{Add}(\kappa, \kappa^{++})$. Assuming GCH it's easy to check that $|\mathbb{Q}| = \kappa^{++}$.
- Since $M^{\kappa^{++}} \subseteq M$, it follows that $j"H \in M$ and so $p_0 = \bigcup j"H$ is a condition in $j(\mathbb{Q})$.
- We can build an $M[j(G)]$ generic filter $K \subseteq j(\mathbb{Q})$ with $p_0 \in K$. Then $\boxed{j"H \subseteq K}$ since K is a filter.

Outline

From the previous section, if κ is κ^{++}-supercompact then there is a forcing extension in which κ is measurable and GCH fails: $2^\kappa = \kappa^{++}$.

Question

Can we obtain a model with a measurable cardinal κ such that $2^\kappa = \kappa^{++}$ from a large cardinal assumption which is weaker than κ^{++}-supercompactness?

How to discover the optimal (consistency strength wise) hypothesis: Suppose κ is measurable and $2^\kappa = \kappa^{++}$. Let $j : V \to M$ be a normal ultrapower embedding by a measure U on κ. Then

$$\kappa^{++} = 2^\kappa \leq (2^\kappa)^M < j(\kappa).$$

Thus, the optimal hypothesis for obtaining a model in which GCH fails at a measurable cardinal is the existence of an embedding $j : V \to M$ with critical point κ such that $M^\kappa \subseteq M$ and $j(\kappa) > \kappa^{++}$.

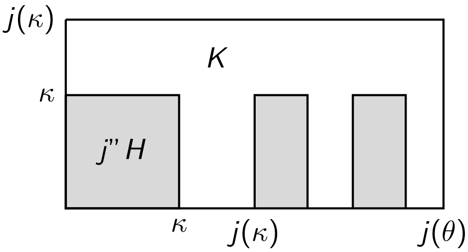
Theorem (Woodin, 80s)

The existence of a measurable cardinal at which GCH *fails is equiconsistent with the existence of an elementary embedding* $j : V \to M$ *with critical point* κ *such that* $M^\kappa \subseteq M$ *and* $j(\kappa) > \kappa^{++}$.

Idea of Proof.

- The forward ($\implies$) direction is an easy *implication* (we did it above).
- For the backward direction...
- The forcing is of the form $\mathbb{P} * \mathbb{Q} = \mathbb{P} * \mathrm{Add}(\kappa, \kappa^{++})$. (This is an over simplification.)
- Suppose $G * H \subseteq \mathbb{P} * \mathbb{Q}$ is V-generic. We lift the embedding to $j : V[G] \to M[j(G)]$ by finding an M-generic filter $j(G) \subseteq j(\mathbb{P})$ with the property $\boxed{j\text{"}G \subseteq j(G)}$.

- To lift j through the $H \subseteq \mathbb{Q}$ forcing, we need a generic $K^* \subseteq j(\mathbb{Q})$ such that $\boxed{j"H \subseteq K^*}$. (There is no master condition.)
- We can obtain a generic $K \subseteq j(\mathbb{Q})$ easily, however, we may not have $j"H \subseteq K$.
- We just change K precisely by performing a type of surgery to obtain K^* satisfying $j"H \subseteq K^*$.
- The amazing thing is that after surgery, K^* is still a generic filter, and this all works. (This is hiding a lot of details.)



- The key to Woodin's proof is that the surgical modifications performed to obtain K^* with $j``H \subseteq K^*$ occur on the range of j.
- This is because if $p \in \mathbb{Q}$ then p has size less than the critical point of j, and so $j(p) = j``p \subseteq \mathrm{range}(j)$.
- Woodin showed, that because the modifications occur on the range of j, the modifications are relatively mild so that K^* remains a generic filter.
- Now we can lift the embedding to the appropriate forcing extension

$$j : V[G][H][K] \to M[j(G)][K^*][j(K)]$$

which shows that κ is a measurable cardinal in $V[G][H][K]$, and that $2^\kappa = \kappa^{++}$ in this model (omitting many details).

Question. What is the consistency strength of having a λ-supercompact cardinal κ such that GCH fails at λ: $2^\lambda = \lambda^{++}$?

We can discover the optimal (consistency strength wise) hypothesis as before.

- Suppose we have κ which is λ-supercompact and $2^\lambda = \lambda^{++}$. Then we have

$$\lambda^{++} = 2^\lambda \leq (2^\lambda)^M < j(\kappa).$$

So the optimal hypothesis is the existence of an embedding $j : V \to M$ with critical point κ such that $M^\lambda \subseteq M$, $j(\kappa) > \lambda^{++}$.

Theorem (C., 2011)

The existence of a cardinal κ that is λ-supercompact such that $2^\lambda = \lambda^{++}$ is equiconsistent with the existence of an elementary embedding $j : V \to M$ with critical point κ such that $M^\lambda \subseteq M$ and $j(\kappa) > \lambda^{++}$.

- The forward direction ($\implies$) is easy as before. For the reverse direction ($\Longleftarrow$)...
- The forcing is of the form $\mathbb{P} * \mathbb{Q} = \mathbb{P} * \mathrm{Add}(\kappa, \lambda^{++})$.
- This means that if $G * H \subseteq \mathbb{P} * \mathbb{Q}$ is V-generic then in $V[G][H]$ we have $2^\kappa = \lambda^{++}$ and then since $\kappa < \lambda$ we have $2^\kappa \leq 2^\lambda$ and in fact $2^\lambda = \lambda^{++}$.
- The reason why this method works is that conditions $p \in \mathbb{Q} = \mathrm{Add}(\kappa, \lambda^{++})$ have size less than the critical point of j. This means $j(p) = j"p$ and Woodin's method of surgery can be adopted to this new case.

Question. Can we obtain a model in which κ is λ-supercompact, GCH holds on the interval $[\kappa, \lambda)$ and $2^\lambda = \lambda^{++}$ from the optimal hypothesis?

Theorem (Friedman and Honzik, 2012)

Suppose GCH holds and there is a $j : V \to M$ with critical point κ such that $M^\lambda \subseteq M$ and $j(\kappa) > \lambda^{++}$. Then there is a cardinal-preserving forcing extension in which the following hold.

- *κ is λ-supercompact.*
- *GCH holds on the interval $[\kappa, \lambda)$.*
- *$2^\lambda = \lambda^{++}$.*

Friedman and Honzik's method does not seem to work for acheiving other behaviors of the continuum function on the interval $[\kappa, \lambda]$ where κ is λ-supercompact.

For example, Friedman and Honzik ask the following question.

Question. Can we obtain a model in which κ is λ-supercompact, and for some γ with $\kappa < \gamma < \lambda$, GCH holds on the interval $[\kappa, \gamma)$ and $2^\gamma = \lambda^{++}$ from the optimal hypothesis?

Theorem (C. and Magidor, 2012)

Suppose GCH *holds and there is a* $j : V \to M$ *with critical point* κ *such that* $M^\lambda \subseteq M$ *and* $j(\kappa) > \lambda^{++}$. *Suppose* γ *is a cardinal with* $\kappa < \gamma < \lambda$. *Then there is a cardinal-preserving forcing extension in which the following hold.*

- κ *is* λ*-supercompact.*
- GCH *holds on the interval* $[\kappa, \gamma)$.
- $2^\gamma = \lambda^{++}$.

Why is something new needed?

- The forcing is of the form $\mathbb{P} * \mathbb{Q} = \mathbb{P} * \mathrm{Add}(\gamma, \lambda^{++})$.
- We let $G * H \subseteq \mathbb{P} * \mathbb{Q}$ be V-generic and lift the embedding as before.
- However, in the key step in the argument, we need to find a generic filter K^* with $j``H \subseteq K^*$, but Woodin's lemma will not work because conditions $p \in \mathbb{Q}$ can have size greater than the critical point. Then $j(p) \neq j``p$.
- This means that when we modify K to obtain K^*, we need to do modifications off of the range of j on the "ghost coordinates" (which is more than before).
- The new key lemma shows that these modifications are still mild.

Lemma (New Key Lemma)

If $B \in M[j(G)]$ and $B \subseteq j(\gamma \times \lambda^{++})$ with $|B|^{M[j(G)]} \leq j(\gamma)$ then

$$\mathcal{I}_B = \{j(d) \cap B \mid d \in P_\gamma(\gamma \times \lambda^{++})^{V[G]}\}$$

has size at most λ in $V[G][H]$.

- This new lemma implies that if we take a condition $p \in K$ and perform surgery to obtain $p^* \in K^*$ (so that $j"H \subseteq K^*$) then p^* is still a condition.
- Thus, $K^* \subseteq j(\mathbb{Q})$ is a filter.
- The new lemma also implies that K^* is $M[j(G)]$-generic.

Indeed, much more can be acheived...

Theorem (C. and Magidor, 2012)

Suppose GCH *holds and there is an elementary embedding* $j : V \to M$ *with critical point* κ *such that for some regular cardinal* $\lambda > \kappa$ *one has*

1. $M^\lambda \subseteq M$,
2. $F : [\kappa, \lambda] \cap \mathrm{REG} \to \mathrm{CARD}$ *is a function satisfying the requirements of Easton's theorem, and*
3. $j(\kappa) > F(\lambda)$.

Then there is a cofinality-preserving forcing extension in which κ *remains* λ*-supercompact and for every regular cardinal* $\gamma \in [\kappa, \lambda]$ *one has* $2^\gamma = F(\gamma)$.

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