What are strong axioms of infinity and why are they useful in mathematics?

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Outline

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Introduction

What are strong axioms of infinity? Or large cardinal axioms?

- There are many degrees of infinity in mathematics:

$$|\mathbb{N}| = \omega = \aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots \aleph_{\aleph_1} < \cdots \aleph_{\aleph_\omega} \cdots$$

- The first infinite cardinal ω has many properties that make it "large" compared to finite numbers $0, 1, 2, \ldots$. For example, if $|X| = n < \omega$ then $|P(X)| < \omega$.
- In a sense ω is a dividing line between the finite and the infinite:

$$\text{“finite”} < \omega \leq \text{“infinite”}$$

- Strong axioms of infinity (or "large cardinal axioms") typically assert that there is a cardinal $\kappa > \omega$ such that the relationship between things smaller than κ and things of size κ is similar to the relationship of "finite" to "infinite."

Cantor and Set Theory

- In the 1870's Cantor's work on convergence of trigonometric series led him to study sets of real numbers and infinitary enumerations.
- 1873: Cantor proved that the set of real numbers $\mathbb{R}$ is uncountable. He showed there there is no 1-to-1 correspondence between $\mathbb{N}$ and $\mathbb{R}$.
- Since every set X must be in 1-to-1 correspondence with one of the $\aleph$'s, Cantor was led to the question: how big is the set of real numbers $\mathbb{R}$?
- The continuum hypothesis CH asserts that $|\mathbb{R}| = \aleph_1$.
- Equivalently, CH asserts that every infinite subset of $|\mathbb{R}|$ is either in 1-to-1 correspondence with $\mathbb{N}$ or with all of $\mathbb{R}$.

First-order logic and ZFC

- There are some conceptual paradoxes in basic set theory, such as Russell's paradox, that led to a precise formulation of first-order axioms for set theory: the axioms of Zermelo-Frankel set theory ZFC.
- First-order logic (through the work of Frege and many others) had already emerged as the main candidate system in which to build a foundational system for mathematics.
- Within the framework of ZFC one can carry out virtually all of mathematics. For example in ZFC, (1) one can build $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$, (2) one can prove the theorems of calculus, functions spaces, topological spaces, abstract algebra, etc.
- Hilbert's Program: Prove that the foundations of mathematics are consistent and complete.

- Proofs in first-order logic are mathematical objects: a proof is a finite sequence of symbols with certain properties.
- We write $T \vdash \varphi$ to mean that there is a proof of φ from the axioms of T.
- Gödel's Completeness Theorem: Logical entailment and provability are equivalent.

$$T \models \varphi \iff T \vdash \varphi$$

The first-order axioms of set theory ZFC:

- (Axiom of infinity) There is an infinite set.
- (The Pairing Axiom)
  $\forall x \forall y \exists z \forall w (w \in z \iff w = x \vee w = y)$
  If x and y are sets then $\{x, y\}$ is a set.
- (The Powerset Axiom)
  $\forall x \exists y \forall z (z \in y \iff \forall w (w \in z \implies w \in x))$
  If x is a set then $P(x)$ is also a set.
- etc.

The axioms of ZFC formulated in first-order logic provided a suitable foundational system for doing mathematics.

Hilbert's Program: Prove that the foundations of mathematics are consistent and complete.

There was a problem...

G•del's Incompleteness Theorem. Suppose T is a consistent recursive set of axioms extending ZF. Then T is incomplete in the sense that there is a sentence φ such that

$$T \nvdash \varphi \text{ and } T \nvdash \neg\varphi$$

G•del's 2nd Incompleteness Theorem. If T is a recursive consistent extension of ZF, then T does not prove $\mathrm{CON}(T)$.

Conclusion. Any recursive set of foundational axioms of mathematics is either inconsistent or incomplete.

Maybe all undecidable propositions in ZFC are "philosophical" like "this statement is not provable" or "ZFC is consistent."

No:

The Continuum Hypothesis (CH), is a natural mathematical assertion that is undecided by ZFC.

More on the Continuum Hypothesis

Fact. Every set X can be put into 1-to-1 correspondence with some cardinal $\aleph_\alpha$.

Fact. $|\mathbb{R}| = |P(\mathbb{N})| = 2^{\aleph_0}$

Question. What is the cardinality of $\mathbb{R}$?
$\omega = \aleph_0, \aleph_1, \aleph_2, \aleph_3, \ldots$?

The Continuum Hypothesis (CH): $|\mathbb{R}| = \aleph_1$ or equivalently

$$\mathrm{CH} : 2^{\aleph_0} = \aleph_1$$

Theorem. (Gödel) If V is model of ZFC then there is a model L of ZFC in which CH is true. $[\mathrm{CON(ZFC)} \implies \mathrm{CON(ZFC+CH)}]$

- Assuming ZFC is consistent, ZFC cannot prove $\neg\mathrm{CH}$.
- L is known as Gödel's constructible universe, obtained recursively by iterating the definable powerset operation. $L_0 = \emptyset$, $L_{\alpha+1} = P_{\mathrm{def}}(L_\alpha)$, $L_\lambda = \bigcup_{\alpha<\gamma} L_\alpha$.

Theorem. (Cohen) If V is a model of ZFC then there is a model $V[G]$ of ZFC in which CH is false. $[\mathrm{CON(ZFC)} \implies \mathrm{CON(ZFC+\neg CH)}]$

- Assuming ZFC is consistent, ZFC cannot prove CH.
- Cohen's proof uses his method of forcing: Starting with a model $V \models \mathrm{ZFC}$ one can choose a partial order $(\mathbb{P}, \leq)$ and use it to obtain a forcing extension $V[G] \supseteq V$ with finely tuned properties, such as $\neg\mathrm{CH}$.

The situation:

Conclusion. If ZFC is consistent then $\mathrm{ZFC} \nvdash \mathrm{CH}$ and $\mathrm{ZFC} \nvdash \neg\mathrm{CH}$.

If ZFC is consistent then it is incomplete, and what's more, there are natural questions that ZFC does not decide, such as CH.

Perhaps there are additional axioms that could be added to ZFC so that in the new theory we could prove $\mathrm{CON}(\mathrm{ZFC})$ and decide CH.

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Strong Axioms of Infinity

Strong Axioms of Infinity and the Continuum Function

Strong axioms of infinity

An inaccessible cardinal κ, is a dividing line like the one between the finite and the infinite, but higher up.

The statement "there is an inaccessible cardinal" is an example of a strong axiom of infinity, or a $\boxed{\textit{large cardinal axiom}}$.

$0, 1, 2, \ldots \omega, \aleph_1, \aleph_2, \ldots \aleph_\omega, \ldots \aleph_{\aleph_1}, \ldots \gamma, \ldots \kappa, \ldots$

Properties of ω: (1) For every $n < \omega$ we have $2^n < \omega$.
(2) If $X \subseteq \omega$ is finite then X is bounded.

De nition. We say that $\kappa > \omega$ is an *inaccessible cardinal* if
(1) for every $\gamma < \kappa$ we have $2^\gamma < \kappa$, and
(2) if $X \subseteq \kappa$ has size $< \kappa$ then X is bounded.

What are large cardinals good for?

Gödel's Second Incompleteness Theorem. ZFC does not prove $\mathrm{CON}(\mathrm{ZFC})$ (assuming ZFC is consistent).

For every sufficient theory, T one has "T does not prove $\mathrm{CON}(T)$."

$$T_0 = \mathrm{ZFC}$$
$$T_1 = \mathrm{ZFC} + \mathrm{CON}(\mathrm{ZFC})$$
$$T_2 = \mathrm{ZFC} + \mathrm{CON}(\mathrm{ZFC}) + \mathrm{CON}(\mathrm{ZFC} + \mathrm{CON}(\mathrm{ZFC}))$$

Fact. The existence of an "inaccessible cardinal" implies $\mathrm{CON}(\mathrm{ZFC})$ (as well as $\mathrm{CON}(T_1)$, $\mathrm{CON}(T_2)$, etc.).

$$T_{\text{inacc}} = \mathrm{ZFC} + \text{"there is an inaccessible cardinal"}$$

$$T_{\text{inacc}} \nvdash \mathrm{CON}(T_{\text{inacc}})$$

Fact. The existence of a “Mahlo cardinal” implies $\mathrm{CON}(\mathrm{ZFC}+$“there is an inaccessible cardinal”$)$.

So by adding strong axioms of infinity to ZFC we increase the strength of our theory.

Assuming ZFC is consistent,

- $\mathrm{ZFC} \nvdash \mathrm{CON}(\mathrm{ZFC})$, but
- $\mathrm{ZFC} + \text{“there is an inaccessible cardinal”} \vdash \mathrm{CON}(\mathrm{ZFC})$

Large cardinals form a linear hierarchy which increases in consistency strength.

0=1
I0—I3
n-huge
huge
almost huge
extendible
supercompact
strongly compact
strong
measurable
Ramsey
Rowbottom
indescribable
weakly compact
Mahlo
inaccessible

Important: The added strength that large cardinals give set theory, is useful in "everyday" mathematics.

Example:
Lebesgue defined a way to measure the size of sets of real numbers.

$m([0,1]) = 1$ (the measure of the interval $[0,1]$ is 1)
$m([3,8]) = 5$
$m([0,1] \cap \mathbb{Q}) = 0$

The function m is called "Lebesgue measure," and is extremely useful in analysis.

Theorem. (Vitali, 1905) The Axiom of Choice (AC) implies there is a set of reals $X \subset \mathbb{R}$ that is not Lebesgue measurable, meaning that $m(X)$ is undefined.

Question. Is the axiom of choice necessary to prove such a set of reals exists?

It seems as though most mathematicians would agree the answer should be YES.

If there is an inaccessible cardinal, then we can prove the answer is YES.

Theorem. (Solovay, 1970) If there is an inaccessible cardinal, then there is a model of $\mathrm{ZF} + \neg\mathrm{AC}$ in which every set of reals is Lebesgue measurable.

Question. Is Solovay's use of an inaccessible cardinal necessary?

Theorem. (Shelah, 1984) YES. If every set of reals is Lebesgue measurable, then $\aleph_1$ is inaccessible in L.

Conclusion. If one wants to be able to be able to prove that the Axiom of Choice is necessary in constructing a non-measurable set, then one MUST accept the consistency of a large cardinal axiom.

Terminology. The theorems of Solovay and Shelah mentioned above show that the statement "every set of reals is Lebesgue measurable" is *equiconsistent* with the existence of an inaccessible cardinal. (This terminology is used because a model satisfying one leads to a model satisfying the other.)

0=1
I0—I3
n-huge
huge
almost huge
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supercompact
strongly compact
strong
$o(\kappa) = \kappa^{++} \longleftrightarrow$ "singular stong limit γ with $2^\gamma > \gamma^+$"
measurable
Ramsey
Rowbottom
indescribable
weakly compact
Mahlo
inaccessible $\longleftrightarrow$ "every set of reals is Lebesgue measurable"

Gödel's Hope: Perhaps adding strong axioms of infinity to ZFC would allow us to decide CH.

To realize Gödel's hope we would like to say something like, "if there is an inaccessible cardinal then CH: $2^\omega = \aleph_1$ holds."

However...

κ
$\vdots$
$\aleph_2$ One can force CH: $2^\omega = \aleph_1$ or $\neg\mathrm{CH}$: $2^\omega = \aleph_2$
$\aleph_1$ with "small" forcing $\mathbb{P}$ that does not affect the
ω fact that κ is inaccessible.
$\vdots$
2
1
0

So if $V \models \mathrm{ZFC}$ + "there is an inaccessible cardinal" then there is a *forcing extension* $V[G]$ such that

$$V[G] \models \mathrm{ZFC} + \kappa \text{ is inaccessible} + \mathrm{CH}$$

and another forcing extension $V[H]$ such that

$$V[H] \models \mathrm{ZFC} + \kappa \text{ is inaccessible} + \neg\mathrm{CH}$$

Essentially what this shows is:

Theorem. (Lévy-Solovay) Small forcing preserves large cardinals, and thus, strong axioms of infinity do not decide CH.

Conclusion. CH is not only independent from ZFC, but also from ZFC + strong axioms of infinity.

Although large cardinal axioms do not fulfill Gödel's hope of deciding CH, they have turned out to be very useful tools for obtaining consistency results.

Slogan: All known statements that one would desire a consistency proof of, can be proven consistent from large cardinals.

Example: there is an axiom called "Martin's Maximum" which can be proven to be consistent from large cardinals, and which decides CH.

$$MM \implies \neg\mathrm{CH} : 2^{\omega} = \aleph_2$$

Idea: Set theorists want to understand the structure of large cardinals—i.e., how they are ordered, their combinatorial properties, etc. Many of the questions about large cardinals are independent (just like CH), and can be controlled by forcing (just like we can control the value of $2^{\aleph_0}$ by forcing).

A major theme in set theory is to study the relationship between large cardinals and forcing. For example, if κ is an inaccessible cardinal in V, and $V[G]$ is some forcing extension, does κ remain inaccessible in $V[G]$?

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CH: $2^\omega = \aleph_1$
GCH: for all cardinals κ, $2^\kappa = \kappa^+$.
We say GCH fails at κ to mean $2^\kappa > \kappa^+$.

Using forcing, Cohen showed that 2^ω can equal any reasonable cardinal $\aleph_1, \aleph_2, \aleph_{17}, \aleph_{\aleph_1}$, etc.

Easton proved a **global** theorem using forcing which says that for every regular cardinal, the value of 2^κ can be anything 'reasonable.'

Theorem. (Easton) In a model $V \models \mathrm{ZFC}$, given any 'reasonable' function F from the regular cardinals to the cardinals, there is a cardinal-preserving forcing extension $V[G]$ in which $2^\kappa = F(\kappa)$ for all regular cardinals κ.

- What is meant by 'reasonable'? It follows from ZFC that for any cardinals $\kappa \leq \lambda$ we have $2^\kappa \leq 2^\lambda$. So, one thing that a 'reasonable' function F must satisfy is that $\kappa \leq \lambda \implies F(\kappa) \leq F(\lambda)$.

Easton's theorem is saying that in ZFC the behavior of the *continuum function* $\kappa \mapsto 2^\kappa$ is highly undetermined.

Note. Suppose κ is an inaccessible cardinal. The forcing used in Easton's theorem to control the global behavior of the continuum function $\kappa \mapsto 2^\kappa$ on the regular cardinals is not small relative to κ. Under certain conditions, one can still prove that Easton's forcing preserves the inaccessibility of κ, but it takes a different argument.

To talk about the way in which large cardinals affect the behavior of the continuum function $\kappa \mapsto 2^\kappa$ we will need a few more types of large cardinals:

De nition. A cardinal κ is called a *measurable cardinal* if there is a κ-complete non-principle ultrafilter U on $P(\kappa)$.

- κ measurable $\implies$ κ is inaccessible
- If κ is measurable then there must be many inaccessible cardinals below κ.

De nition. A cardinal κ is λ*-supercompact* $(\kappa < \lambda)$ if there is a "certain type" of κ-complete ultrafilter.

- If κ is λ-supercompact $(\kappa < \lambda)$ then κ is measurable and there are many measurable cardinals below κ.

κ is *supercompact* if it is λ-supercompact for all $\lambda > \kappa$.

Although large cardinals do not decide CH, they do imply certain things about possible behaviors of the *continuum function* $\kappa \mapsto 2^\kappa$.

Theorem. (Scott) If κ is a measurable cardinal and $2^\kappa > \kappa^+$ then for many regular cardinals $\gamma < \kappa$ we must have $2^\gamma > \gamma^+$.

- We say that the failure of GCH at κ ($2^\kappa > \kappa^+$), reflects below κ ($2^\gamma > \gamma^+$ for many $\gamma < \kappa$).
- In particular, GCH cannot first fail at a measurable cardinal.

Theorem. If GCH holds below a supercompact cardinal κ, then GCH must hold at every cardinal—in other words, if $2^\gamma = \gamma^+$ for all $\gamma < \kappa$ then it must be the case that $2^\gamma = \gamma^+$ for all cardinals (even those greater than κ).

Fact 1. Easton showed that the continuum function $\kappa \mapsto 2^\kappa$ can be forced to equal any ‘reasonable’ function on the regular cardinals.

Fact 2. The existence of large cardinals restricts the possible behaviors of the continuum function $\kappa \mapsto 2^\kappa$.

Main Question. Given a large cardinal κ, what Easton functions can be forced to equal the continuum function *while preserving the large cardinal property of κ?*

Easton's Theorem and Large Cardinals

Theorem. (Menas, 1976) If κ is a supercompact cardinal and F is any *locally definable* Easton function, then there is a forcing extension in which κ remains supercompact and $2^\kappa = F(\kappa)$ for each regular cardinal κ.

Theorem. (Friedman-Honsik, 2008) If κ is a strong cardinal and F is any *locally definable* Easton function, then there is a forcing extension in which κ remains strong and $2^\kappa = F(\kappa)$ for each regular cardinal κ.

Theorem. (C., 2012) If κ is a Woodin cardinal and F is **any Easton function with** $F``\kappa \subseteq \kappa$, then there is a forcing extension in which κ remains Woodin and $2^\kappa = F(\kappa)$.

- $F``\kappa \subseteq \kappa$ means the for every $\gamma < \kappa$ we have $F(\gamma) < \kappa$.
- This theorem says that in a certain sense the existence of a Woodin cardinal (a type of strong axiom of infinity) has little impact on the possible behaviors of the continuum fundtion $\kappa \mapsto 2^\kappa$.

The Failure of GCH at a Degree of Supercompactness

Theorem. (Woodin) The existence of a measurable cardinal κ such that $2^\kappa = \kappa^{++}$ is *equiconsistent* with the existence of a cardinal κ that is κ^{++}-tall.

- Remember "equiconsistent" means a model of one leads to a model of the other.
- This theorem says that obtaining a model in which there is a measurable cardinal κ such that $2^\kappa > \kappa^+$ (GCH fails at κ) requires an additional large cardinal assumption beyond that of a simple measurable cardinal.
- For the backward direction, the idea is: start with a model V in which κ has the large cardinal property of being κ^{++}-tall, and then move to a forcing extension $V[G]$ in which $2^\kappa = \kappa^{++}$ and <u>κ remains measurable</u>.

Theorem. (C., 2010) The existence of a cardinal κ that is λ-supercompact (where $\kappa < \lambda$) such that $2^\lambda = \lambda^{++}$ is equiconsistent with the existence of a λ-supercompact cardinal κ that is also λ^{++}-tall.

0,1,2... ω, $\aleph_1$, ... κ, κ^+, ... λ, λ^+, λ^{++}, ...

$$2^\kappa = \lambda^{++} \implies 2^{\kappa^+} = \lambda^{++} \implies 2^\lambda = \lambda^{++}$$

- The theorem is proved by using forcing to make $2^\kappa = \lambda^{++}$, which then implies that $2^\lambda = \lambda^{++}$ since $\kappa \leq \lambda$.
- This suggests the question, can one prove a similar theorem to obtain a model in which κ is λ-supercompact and $2^\kappa = \kappa^+$ and $2^\lambda = \lambda^{++}$?

Theorem. (Friedman-Honzik, 2012) If κ is λ-supercompact and λ^{++}-tall then there is a forcing extension in which κ is λ-supercompact, $2^\kappa = \kappa^+$, and $2^\lambda = \lambda^{++}$.

The natural question to ask now is, can one prove a similar theorem but force any 'reasonable' behavior (in the sense of Easton) of the continuum function $\gamma \mapsto 2^\gamma$ on the interval $[\kappa, \lambda]$?

Theorem. (Cody-Magidor, 2012)
Suppose $F : [\kappa, \lambda] \cap \mathrm{REG} \to \mathrm{CARD}$ is a function that is a 'reasonable' candidate for the continuum function (in the sense of Easton). If κ is λ-supercompact and $F(\lambda)$-tall, then there is a forcing extension in which κ is λ-supercompact and the continuum function on the interval $[\kappa, \lambda]$ can have any 'reasonable' behavior.

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