SELECTED SOLUTIONS FOR COMPUTABILITY 2019

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Exercise 4.2.9. *(1) Show that every infinite, finitely branching tree T has a path recursive in T''.*
(2) Build a recursive tree containing at least one path and every path of it computes $0''$.

Proof. (1) Given a infinite, finitely branching tree T. We define a $\Delta^0_3(T)$ path P so that $P \leq_T T''$. Let

$$P = \{\sigma \in \mathbb{N}^{<\mathbb{N}} \mid \exists f \in (\mathbb{N}^{<\mathbb{N}})^{|\sigma|+1} \big(\varphi(f) \wedge f(|\sigma|) = \sigma\big)\},$$

Note that $f \in (\mathbb{N}^{<\mathbb{N}})^{|\sigma|+1}$ means $f : |\sigma| + 1 \to \mathbb{N}^{<\mathbb{N}}$, so it is a $|\sigma| + 1$ long sequence of finite sequence of natural numbers. Note also that $\exists f \in (\mathbb{N}^{<\mathbb{N}})^{|\sigma|+1}$ is not a bounded quantifier. We would expect $\varphi(f)$ saying that f is a sequence telling us how to inductively choose an infinite path. Explicitly, $\varphi(f)$ can be written as

$$f(0) = \langle\rangle \wedge \forall n + 1 \in \operatorname{dom} f \exists k \in \operatorname{ran}(\operatorname{ran} f)\big(f(n+1) = f(n)^\frown\langle k\rangle \wedge \psi(k, f, n)\big).$$

where $\psi(k, f, n)$ roughly says that k is the *least* choice so that the subtree above $f(n)^\frown\langle k\rangle$ is still an infinite tree. More explicitly, $\psi(k, f, n)$ is

$$\begin{aligned}&\forall l \exists \tau \in \mathbb{N}^l (\tau \in T \wedge \tau{\restriction}(n+2) = f(n+1))\\ &\qquad \wedge \forall k' < k\big(f(n)^\frown\langle k'\rangle \in T \to \exists l \forall \tau \in \mathbb{N}^l(\tau \in T \to \tau{\restriction}(n+2) \neq f(n)^\frown\langle k'\rangle)\big).\end{aligned}$$

Note that the quantifiers in red are unbounded. Therefore P can be defined by a $\Sigma^0_3(T)$ formula. Since it can be proved that such a sequence f is unique, P can also be defined as

$$P = \{\sigma \in \mathbb{N}^{<\mathbb{N}} \mid \forall f \in (\mathbb{N}^{<\mathbb{N}})^{|\sigma|+1} \big(\varphi(f) \to f(|\sigma|) = \sigma\big)\}.$$

In this case P is also $\Pi^0_3(T)$, hence $\Delta^0_3(T)$.

This a proof of (2) given by Liu Yong.

IDEA:

We want to construct a tree containing at least one path and whose every path P satisfying.

$$P(e) = \begin{cases} \text{the starting stage for } P{\restriction}e, & \text{if } \Phi^{0'}_e(e)\uparrow, \\ t \ (>\text{the starting stage for } P{\restriction}e), & \text{if } \Phi^{0'}_e(e)\downarrow. \end{cases} \tag{4.2.9.1}$$

The *starting stage* for $P{\restriction}e$ is a number $s \geq e$ serving two purposes. First, it indicates that eventually $\Phi^{0'}_e(e)\uparrow$; second, if for some $d < e$, $P(d)$ is landing at its own starting stage claiming that Φ_d will diverge, then s is informative (big) enough to see an evidence that it is right to make that claim.

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CONSTRUCTION: First, we define $s_\sigma \in \mathbb{N}$ being the starting stage for $\sigma \in \mathbb{N}^{<\mathbb{N}}$ inductively as follow: Assume $|\sigma| = e$, s_σ is the *starting stage for* σ if and only if s_σ is the least $\geq e$ such that for every $d < e$, if $\sigma(d)$ is the starting stage (indicating that Φ_d will diverge), then there is $t \in [e, s_\sigma]$ witnessing that $\Phi^{0'_t}_{d,t}(d)\uparrow$. Note that $\{\langle \sigma, s\rangle \mid s \text{ is the starting stage for } \sigma\}$ is recursive.

Now, fix $\sigma \in \mathbb{N}^{<\mathbb{N}}$ such that $|\sigma| = e$, and $s \in \mathbb{N}$. Assume inductively that we know $\sigma \in T$. We want to see if $\sigma^\frown s \in T$.

(a) If there is $d < e$ such that $\sigma(d) >$ the starting stage for $\sigma\restriction d$, $\Phi^{0'_{\sigma(d)}}_{d,\sigma(d)}(d)\downarrow$, and $e > \sigma(d)$ is the least such that $\Phi^{0'_e}_{d,e}(d)\uparrow$, then $\sigma^\frown s \notin T$. Note that this decision is made regardless of s. The idea is that we exam to the step e, as soon as we find some previous converged Φ_d diverges, we stop any extension recording a $< e$ *down stage* for level d.

(b) $\sigma^\frown s \in T$ only if s is either the starting stage for σ or a down stage $>$ the starting stage. Note that if there is some $d < e$ such that $\sigma(d)$ is the starting stage while we cannot find a witness of diverge in $[e, s_\sigma]$, then $\sigma^\frown s \notin T$.

This completes the construction of T.

VERIFICATION:

(i) T is a recursive tree by construction.

(ii) Let P be such that

$$P(e) = \begin{cases} \text{the starting stage for } P\restriction e, & \text{if } \Phi^{0'}_e(e)\uparrow, \\ t, & \text{if } t > \text{the starting stage for } P\restriction e, \\ & \Phi^{0'}_{e,t}(e)\downarrow \text{ and } 0'_t\restriction(\text{use}\,\Phi^{0'}_{e,t}(e)) \prec 0'. \end{cases}$$

Then P will be a path on T. Note first that such a path won't be killed by (a). The starting stages are all well defined, namely cannot be infinite. Finally, show that it won't be killed because of (b) in any cases.

(iii) Any path P on T must satisfies (4.2.9.1). By (b), $P(e)$ can only be the starting stage for $P\restriction e$, or a down stage.

Assume to a contradiction that $\Phi^{0'}_e(e)\downarrow$ while $P(e)$ is the starting stage for $P\restriction e$. Then for some $t > e$, $\Phi^{0'_t}_{e,t}(e)$ will stay at converge afterwards, and for any $d \geq t$, we can find no witness for the claiming of divergence, no starting stage for $P\restriction d$ (or the starting stage being ∞), and so no extension of $P\restriction d$ in T.

Now, assume $P(e) = t$ is a down stage, namely $\Phi^{0'_t}_{e,t}(e)\downarrow$. If eventually $\Phi^{0'}_e(e)\uparrow$, then it will be killed by (a) at some stage.

(iv) $0'' \leq_T P$. To see if $e \in 0''$, check if $P(e)$ is the starting stage for $P\restriction e$.

This another proof of (2) given by Wang Sa.

IDEA:

Recall that we can construct a finitely branching recursive tree T^0 containing a unique path $P \equiv_T 0'$. By relativization, we can construct a finitely branching tree $T^A \leq_T A$ containing a unique path $P \equiv_T A'$. For a string $\sigma \prec A$, we can compute a finite subtree T^σ of an initial segment of T^A by making sure the use of A not exceeding $|\sigma|$.

We construct a infinite branching recursive tree T^* so that its even levels recording the construction of T^0, while the odd levels using a node σ of T^0 to compute an finite subtree of an initial segment of T^σ and a cofinal path of T^σ.

In the end, the even part of a path of T^* would be the path P of T^0 and $P \equiv_T 0'$; the odd part of T^* recording the construction of a path of T^P, which computes $P' \equiv_T 0''$.

CONSTRUCTION:

By induction, we assume $\rho \in T^*$ and try to decide if a successor of ρ is in T^*.

If $|\rho|$ is even. We put $\rho^\frown\{\sigma\}$ into T^*, only if $\sigma \in T^0$ and for any even $i < |\rho|$, $\rho(i) \prec \sigma$.

If $|\rho| = 2e+1$ is odd. We use $\sigma = \rho(e)$ as oracle. We put $\rho^\frown\{\langle t, p\rangle\}$ into T^*, only if it satisfies

(a) t is a finite subtree of an initial segment of T^σ (or T^A for any $A \succ \sigma$) whose height is e,
(b) p is a "path" on t of length e,
(c) for any odd $i < |\rho|$, if $\rho(i) = \langle t_i, p_i\rangle$, then $t_i \subsetneq t$ and $p_i \subsetneq p$.

VERIFICATION:

(a) T^* is recursive by construction.

(b) We show that there exists a path on T^*. Let P_0 be the unique path of T^0, and P_1 be the unique path of T^{P_0}. We can define a path P on T^* as follow. Given $\sigma \prec P$ defined, assume that $|\sigma| = 2e$. We define $P(2e) =$ the shortest $\sigma \prec P_0$ such that σ computes a finite subtree $t \subset T^{P_0}$ containing $P_1\upharpoonright e$ as a cofinal "path". Then we let $P(2e+1) = \langle t, P_1\upharpoonright e\rangle$, while t is the smallest such finite subtree.

(b) If P is a path on T^*, then the even part of P is the unique path P_0 of T^0. For the odd part, by monotonicity, $\bigcup_{i \text{ is odd}} t_i$ is an infinite tree and $\bigcup_{i \text{ is odd}} p_i$ is an infinite path on that tree. Assume to contradiction that $\bigcup_i t_i \nsubseteq T^{P_0}$, then some $t_i \nsubseteq T^{P_0}\upharpoonright\frac{i-1}{2}$. For long enough σ, we can assume $T^{P_0}\upharpoonright\frac{i-1}{2} = T^\sigma\upharpoonright\frac{i-1}{2}$. Therefore no subtree of $T^\sigma\upharpoonright\frac{i-1}{2}$ extends t_i, and so $\langle t_i, p_i\rangle$ won't be on an infinite path. A contradiction. Now, $\bigcup_i t_i$ is an infinite subtree of T^{P_0} and $\bigcup_i p_i$ is a path on it. Since P_1 is the unique path of T^{P_0}, $\bigcup_i p_i = P_1$ as desired.

(c) To compute $0''$ from a path P of T^*. Since $P_1 = \bigcup_i p_i$ computes $0''$, we just look for a large enough $\langle t, p\rangle$ on P and compute with oracle p.

$\square$

THIS IS A WRONG PROOF of 4.2.9 (2)! We build a recursive but possibly infinitely branching tree such that it contains exactly one path, which computes $0''$. We hope the path P would satisfies

$$P(e) = t, \tag{4.2.9.2}$$

where t is the last "change point", namely, t is the least such that either for all $s \geq t$, $\Phi_{e,s}^{0'_s}(e)\downarrow$ and so $\Phi_e^{0'}(e)\downarrow$, or for all $s \geq t$, $\Phi_{e,s}^{0'_s}(e)\uparrow$ and so $\Phi_e^{0'}(e)\uparrow$.

Note that such a P encodes redundant information just to compute $0''$. To see if $e \in 0''$, just compute up to $\Phi_{e,P(e)}^{0'_{P(e)}}(e)$, and see if it converges.

IDEA:

When we are trying to compute every $\Phi_{e,s}^{0'_s}(e)$, we only allow the tree to grow only upon those recent "change points".

CONSTRUCTION:

Given a $\sigma \in \mathbb{N}^{<\mathbb{N}}$. Let $s = \max(\{|\sigma|\} \cup \operatorname{ran}\sigma)$ be the step we would like to check up to. We define $\sigma \in T$ if and only if for all $e < |\sigma|$, $\sigma(e) = t$, where t is the least such that either for all $t \leq s' \leq s$, $\Phi_{e,s'}^{0'_{s'}}(e)\downarrow$, or for all $t \leq s' \leq s$, $\Phi_{e,s'}^{0'_{s'}}(e)\uparrow$.

<u>VERIFICATION</u>:

(a) T is a tree. Given $\tau \in T$ and $\sigma \subset \tau$. For τ, we need to check every point up to, say s^τ steps; for σ, say s^σ steps. Clearly, $s^\sigma \leq s^\tau$, but still no smaller than every $\sigma(e)$. Therefore, every $\sigma(e)$ remains to be the last change point before $s^\sigma \leq s^\tau$, so $\sigma \in T$.

(b) T is recursive by definition.

(c) The path P defined in (4.2.9.2) is a path on T. For any n, we have to check for every $e < n$ if $P(e)$ is the last change point up to some step $s^n \geq P(e)$, but it must be, since there will no change point every . NO! If finally $\Phi_e^{0'}(e)\uparrow$, it might be the case that $\Phi_{e,s}^{0'_s}(e)$ converge, diverge,... for infinitely many times. Then there will be no path!

Here is another WRONG approaching! If we just want to make

$$P(e) = \begin{cases} 0, & \text{if } \Phi_e^{0'}(e)\uparrow, \\ t+1, & \text{if } \Phi_{e.t}^{0'}\downarrow. \end{cases}$$

Then the construction will fail: either the tree is not recursive (or not a tree at all), or there will be a wrong path. WHY we need to set the starting stage in the right proof?

Exercise 4.3.17. *Show that*

(1) there is a set A such that $A \leq_T 0'$ but $A \not\leq_{wtt} 0'$;
(2) there are sets A, B such that $A \leq_{wtt} B$ but $A \not\leq_{tt} B$.

Proof. For (1).

We build a set A such that $A \leq_T 0'$, but $A \not\leq_{wtt} 0'$. For each $n = \langle e, i\rangle$, let

$$A_s(n) = \begin{cases} 1, & \text{if } \Phi_{e,s}^{0'_s}(n)\downarrow = 0 \text{ and } \varphi_{i,s}(n)\downarrow > \text{use}\,\Phi_{e,s}^{0'_s}(n) \\ 0, & \text{otherwise.} \end{cases}$$

<u>VERIFICATION</u>:

(a) Clearly, $\langle A_s\rangle_{s\in\mathbb{N}}$ is recursive.

(b) Fix any $n = \langle e, i\rangle$. If $\Phi_e^{0'}(n)\downarrow$, then both $\Phi_{e,s}^{0'_s}(n)$ and use $\Phi_{e,s}^{0'_s}(n)$ converges with respect to $s \to \infty$, and so $\lim_{s\to\infty} A_s(n)\downarrow$. Else if $\Phi_e^{0'}(n)\uparrow$. If it is the case that $\Phi_{e,s}^{0'_s}(n)$ converge at the statue $\uparrow$ (with respect to $s \to \infty$), then $\lim_{s\to\infty} A_s(n)\downarrow = 0$. It is also possible that $\Phi_{e,s}^{0'_s}(n)\downarrow$ for infinitely many times but still $\Phi_e^{0'}(n)\uparrow$. In this case, we must have the negation of $\varphi_{i,s}(n)\downarrow >$ use $\Phi_{e,s}^{0'_s}(n)$ for cofinite many s, and so $\lim_{s\to\infty} A_s(n)\downarrow = 0$. Therefore, $\lim_{s\to\infty} A_s(n)$ always converges, and so $A \leq_T 0'$.

(c) Note that the only case when $\lim_{s\to\infty} A_s(n)\downarrow = 1$ is both $\Phi_e^{0'}(n)\downarrow = 0$ and $\varphi_i(n)\downarrow >$ use $\Phi_e^{0'}(n)$. Therefore, A diagonalized every possibility of $A \leq_{wtt} 0'$. □

Proof. For (2).

This proof uses finite injury priority method.

<u>IDEA</u>: Recall that $A \leq_{tt} B$ if and only if there is a total functional Φ_e (i.e. Φ_e^X is total for any X) such that $A = \Phi_e^B$. We build A and B together to satisfy the requirements

$$R_e: \quad \text{If } \Phi_e^X \text{ is total for all } X, \text{ then } A \neq \Phi_e^B.$$

We also need to code enough information in B so that $A \leq_{wtt} B$.

<u>CONSTRUCTION</u>:

Step 0: Let $A_0 = B_0 = \emptyset$, all requirements R_e are unsatisfied, the boundaries for protecting what we have done for requirements of indices $\leq e$ is set to be $b_{e,0} = 0$ for every e.

Step s+1: Assume A_s and B_s are defined. Search for the first $\langle e, \sigma, t\rangle$ such that R_e is currently marked as unsatisfied, σ is a finite 01-string such that

$$(B_s{\restriction}b_{e,s})^\frown\{1\}^{e+1} \prec \sigma \prec (B_s{\restriction}b_{e,s})^\frown\{1\}^{e+1\frown}\{0\}^{\mathbb{N}},$$

and

$$\Phi^{\sigma}_{e,t}(b_{e,s}+e)\downarrow.$$

$x = b_{e,s}+e$ is where we want to diagonalize for R_e.

Note that since there are infinitely many unsatisfied total functional Φ_e, we can always find such $\langle e, \sigma, t\rangle$. Now, assume e and σ are what we find.

Let $b_{i,s+1} = b_{i,s}$ for $i < e$ (we keep the boundaries untouched, since we would not injury efforts for these requirements), and

$$b_{j,s+1} = \max\{\mathrm{use}\,\Phi^{\sigma}_e(x), b_{e,s}+e+2\}$$

for all $j \geq e$ (we set the boundary for satisfied R_e. All R_j for $j > e$ become unsatisfied, so no additional protection needed).

We set $A_{s+1}, B_{s+1} \in 2^{b_{e,s+1}}$ as follow. Let $B_{s+1} \preceq \sigma$. Namely, we attach $e+1$ many 1' after $B_s{\restriction}b_{e,s}$ and some (at least one) 0's enough to be used as oracle in the computation $\Phi^{\sigma}_e(x)$. Let $A_{s+1}(x) = 1 - \Phi^{\sigma}_e(x)$, and $A_{s+1}(y) = 0$ for $y \neq x$ and $y \geq b_{e,s}$.

We mark R_e as satisfied, and R_j *unsatisfied* for all $j > e$.

We define A and B such that $A(x) = \lim_{s\to\infty} A_s(x)$ and $B(x) = \lim_{s\to\infty} B_s(x)$.

<u>VERIFICATION</u>:

By the construction, efforts for requirement R_e can only be injured when we act for requirements R_i $(i < e)$.[1] Therefore, for each e the efforts for R_e can only be injured finitely many times. Now, fix an e such that Φ_e is a total functional. There is a time, say s, when there will be no more acts for any R_i for $i < e$. The intended diagonalization point $b_{e,s}+e$ is now fixed. Since Φ_e is a total functional, we are guaranteed to find the right σ. Therefore, R_e will eventually be satisfied. Hence, $A \not\leq_{tt} B$.

To see that $A \leq_{wtt} B$ witness by the function $x \mapsto x+2$. For each x, to compute $A(x)$, we first look at $B(x)$ and $B(x+1)$. If they are not 10, then $A(x) = 0$. If they are, we count the number of continuous 1's starting from $B(x)$ backward, say it is $e+1$. Then we try to compute $\Phi^{\sigma}_e(x)$ for some σ extending $B{\restriction}(x+1)$ only by 0's. Such a σ will be found. By our construction $\Phi^B_e(x)\downarrow$ and the computation uses an initial segment of B of the form of such σ. Then $A(x) = 1 - \Phi^{\sigma}_e(x) = 1 - \Phi^B_e(x)$. Note that we have only used information from $B{\restriction}(x+2)$. □

Exercise 4.5.6. REC $= \{e \in \mathbb{N} \mid W_e$ *is recursive* $\}$ *is 1-complete for* Σ_3 *class.*

Proof. REC is Σ_3 because W_e is recursive if and only if

$$\exists d \forall n \big[(\forall s \Phi_{e,s}(n)\uparrow \to \exists t \Phi_{d,t}(n)\downarrow = 0) \wedge (\exists u \Phi_{e,u}(n)\downarrow \to \exists v \Phi_{d,v}(n)\downarrow = 1)\big],$$

which is equivalent to

$$\exists d \forall n \forall u \exists s \exists t \exists v \big[(\Phi_{e,s}(n)\uparrow \to \Phi_{d,t}(n)\downarrow = 0) \wedge (\Phi_{e,u}(n)\downarrow \to \Phi_{d,v}(n)\downarrow = 1)\big].$$

[1] We always have $b_{i,s} \leq b_{j,s}$ for any $i < j$ and any s. At the end of each successor step, both $A_{s+1}{\restriction}b_{e,s} = A_s{\restriction}b_{e,s}$ and $B_{s+1}{\restriction}b_{e,s} = B_s{\restriction}b_{e,s}$, so our act does not injure the efforts for R_i $(i < e)$.

The latter one is Σ_3

Now, given any Σ_3 set X. We can assume $X = \{x \mid \exists y R(x,y)\}$ where R is a Π_2 relation. Since inFIN $= \{e | W_e \text{ is infinite }\}$ is 1-complete for Π_2 class, there is a 1-1 recursive function g such that $R(x,y)$ if and only if $W_{g(x,y)}$ is infinite.

We define a 1-1 recursive function $f : \mathbb{N} \to \mathbb{N}$ so that for each $x \in \mathbb{N}$, $W_{f(x)}$ is an r.e. set generated as follow.

We will try to diagonalize to make $\overline{W_{f(x)}}$ not r.e. That is we hope it will meet the requirements:

$$R_y : \quad \overline{W_{f(x)}} \neq W_y.$$

At step 0, we let $\overline{W_{f(x),0}} = \mathbb{N}$ and so $W_{f(x),0} = \emptyset$, the movable marker Γ_y is located at $b^y_{x,0} = y$ for every y.

At step $s+1$. Assume $\overline{W_{f(x),s}} = \{b^0_{x,s} < \cdots < b^y_{x,s} < \cdots\}$, namely the movable marker Γ_y is located at $b^y_{x,s}$ for each y.

For s is even, as before, for each $y \leq s$, if we find $W_{g(x,y),s} \neq W_{g(x,y),s+1}$, we enumerate $b^y_{x,s}$ from $\overline{W_{f(x),s}}$ into $W_{f(x),s+1}$. By doing this, we move the maker Γ_i (for $i \geq y$) from $b^i_{x,s}$ to $b^{i+1}_{x,s}$. Note that we only have to act for finitely many $y \leq s$. We can assume we act at most once at each step, say for y, in which case the movable marker Γ_i will not move (namely $b^i_{x,s+1} = b^i_{x,s}$) if $i < y$, or move to $b^i_{x,s+1} = b^{i+1}_{x,s}$ if $i \geq y$. This is how $\overline{W_{f(x),s+1}}$ is defined. If no such y, we do nothing.

For s is odd, we look for $y \leq s$ such that there is (we can assume a unique) $n \in W_{y,s+1} - W_{y,s}$, $n \geq b^y_{x,s}$ and R_y has never been act for. If we found, we enumerate all numbers in $[b^y_{x,s}, n]$ into $W_{f(x),s+1}$, namely, we move the makers Γ_i $(i \geq y)$ (finitely rightward) to the location $b^i_{x,s+1} = b^{i+k}_{x,s}$ where k is the least such that $b^{y+k}_{x,s} > n$. If no such y, we do nothing.

Verification.

If $x \in X$, then there exists y such that $W_{g(x,y)}$ is infinite, so $\lim_{s\to\infty} b^i_{x,s} \uparrow$ for all $i \geq y$, so $\overline{W_{f(x)}} = \{\lim_{s\to\infty} b^i_{x,s} \mid i \in \mathbb{N}\}$ is finite and $W_{f(x)}$ is cofinite, and so both recursive, and so $f(x) \in \text{REC}$.

If $x \notin X$, i.e. $\forall y\ W_{g(x,y)}$ is finite and so the marker Γ_y would only be moved rightward finitely many times at odd steps. Since every requirement R_y is act for at most once and a marker Γ_i is only moved rightward at even step when we act for requirements R_y for $y \leq i$. In other word, each Γ_i is moved rightward no more than i times at even steps. Therefore $\lim_{s\to\infty} b^y_{x,s} \downarrow$ for every y, so $\overline{W_{f(x)}}$ is infinite. Moreover, for every y such that W_y *is infinite*, R_y must be act for at sometime (since every $b^y_{x,s}$ converges), so that there is $n \in W_y$ but $n \notin \overline{W_{f(x)}}$ witnessing that $\overline{W_{f(x)}} \neq W_y$. Therefore $\overline{W_{f(x)}}$ is not r.e. and $f(x) \notin \text{REC}$ as desired. □

This is another proof given by Wang Sa.

Proof. As we have shown that COF$= \{e \in \mathbb{N} \mid W_e \text{ is cofinite }\}$ is 1-complete for Σ_3 class, it suffices to show REC $\leq_1$ COF.

By s-m-nTheorem, there is a 1-1 function f such that for each e, $f(e)$ is the index of the following r.e. set:

$$W_{f(e)} = \{\langle i,j\rangle \mid (i \leq j) \wedge (i \in 0' \vee j \in W_e)\}$$

We show that $W_{f(e)}$ is recursive if and only if W_e is cofinite.

Assume W_e is not cofinite. We show $W_{f(e)}$ computes $0'$, and so is not recursive. Let x be any number such that $x \notin 0'$, or we can just assume $0 \notin 0'$. Note that x is a fixed number, which can be built into a program. To see if $i \in 0'$. We wait until finding some $j \in W_e$ such that $j \geq i$ and $j \geq x$, and moreover $\langle x, j\rangle \notin W_{f(e)}$. Such j can always be find, since otherwise either W_e or $0'$ is cofinite. Now, $i \in 0'$ if and only if $\langle i, j\rangle \in W_{f(e)}$.

Now, assume W_e is cofinite. Then there are only finitely many $\langle i, j\rangle$ such that $i \leq j$ and $\langle i, j\rangle \notin W_{f(e)}$. Build this information in the program. To decide if $\langle i, j\rangle \in W_{f(e)}$. First, check if $i \leq j$. If not, return "no"; if so ask the "build in oracle". □

Exercise 5.1.29. *Every uppersemilattice (usl) $\mathcal{L}$ is locally countable, namely, for any finite $F \subset \mathcal{L}$, the subusl $\mathcal{F}$ of $\mathcal{L}$ generated by F is finite. Moreover, there is a uniform recursive bound on $|\mathcal{F}|$ that depends only on $|\mathcal{F}|$.*

Proof. Given u.s.l. $\mathcal{L}$ and a finite set $F \subset \mathcal{L}$. For each $A \subset F$, let $\underline{A} \in \mathcal{L}$ be the least upper bound of A in $\mathcal{L}$. Then $\left\{\underline{A} \;\middle|\; A \in P(F)\right\}$ is the sub u.s.l. of $\mathcal{L}$ generated from F, and its cardinality is less or equal to $2^{|F|}$. □

Exercise 5.1.30. *Given finite uppersemilattices $\mathcal{F} \subset \mathcal{L}$ and an usl extension $\hat{\mathcal{F}}$ of $\mathcal{F}$ generated over $\mathcal{F}$ by one new element (with $\hat{\mathcal{F}} \cap \mathcal{L} = \mathcal{F}$). Prove that $\hat{\mathcal{F}}$ is finite, and there is a finite usl extension $\hat{\mathcal{L}}$ of $\mathcal{L}$ containing $\hat{\mathcal{F}}$.*

Proof. Given finite u.s.l. $\mathcal{F} \subset \mathcal{L}$. Let $\hat{\mathcal{F}} \supset \mathcal{F}$ be any finite extension. We need to find a procedure to produce an common extension $\hat{\mathcal{L}}$ of $\hat{\mathcal{F}}$ and $\mathcal{L}$ from $\hat{\mathcal{F}}$ and $\mathcal{L}$ effectively.

First, we define the relation $\leq_0$ on $\hat{\mathcal{F}} \cup \mathcal{L}$ as follow: for $a \in \hat{\mathcal{F}} - \mathcal{L}$, and $b \in \mathcal{L} - \mathcal{F}$, $a \leq_0 b$ if and only if there is $c \in \mathcal{F}$ such that $a \leq_{\hat{\mathcal{F}}} c$ and $c \leq_{\mathcal{L}} b$; while $b \leq_0 a$ if and only if there is $c \in \mathcal{F}$ such that $b \leq_{\mathcal{L}} c$ and $c \leq_{\hat{\mathcal{F}}} a$. To check $\leq_0$ is transitive, assume $u \leq_0 v$ and $v \leq_0 w$, we should look at all cases of u, v, w being in $\hat{\mathcal{F}} - \mathcal{L}$ or $\mathcal{L}$.

Second, we add a greatest element to $(\hat{\mathcal{F}} \cup \mathcal{L}, \leq_0)$ if there is yet none, and the order relation accordingly.

Clearly, the procedure is effective, and the final structure is a finite u.s.l. extending both $\hat{\mathcal{F}}$ and $\mathcal{L}$.

An alternative solution proposed by Liu, Mingjun:

Given $\mathcal{L}$ and $\hat{\mathcal{F}}$. First, define the partial order $\leq_0$ on $\mathcal{L} \cup \hat{\mathcal{F}}$ as usual. Then define $\mathcal{P} = P(\mathcal{L} \cup \hat{\mathcal{F}})$, and define the relation as follow.

$$X \leq_{\mathcal{P}} Y \text{ if and only if } X \supset Y.$$

For $a \in \mathcal{L} \cup \hat{\mathcal{F}}$, let $X_a = \left\{b \in \mathcal{L} \cup \hat{\mathcal{F}} \;\middle|\; a \leq_0 b\right\}$. Claim: (i) $(\mathcal{P}, \leq_{\mathcal{P}})$ is a u.s.l.; (ii) $a \mapsto X_a$ is an embedding of $\mathcal{L} \cup \hat{\mathcal{F}}$ into $\mathcal{P}$. □

Exercise 5.1.31. *Prove that there is a recursive usl $\mathcal{L}$ such that every countable usl can be embedded in it as an usl.*

Proof. We construct a universal countable u.s.l. $\mathcal{L} = \bigcup_s \mathcal{L}_s$ as follow. We always keep the domain of $\mathcal{L}_s$ a finite initial segment of $\mathbb{N}$.

Let $\mathcal{L}_0 = \emptyset$.

Given a finite u.s.l. $\mathcal{L}_s$ defined, we should define $\mathcal{L}_{s+1}$. Let $\{\mathcal{F}_i\}_{i<k}$ lists all the one element extension of a sub u.s.l. of $\mathcal{L}_s$ modula isomorphism. By *one element*

extension, we mean a u.s.l. $\hat{\mathcal{F}}$ generated from $\mathcal{F} \cup \{a\}$ where $\mathcal{F}$ is a sub u.s.l. of $\mathcal{L}_s$ and $a \notin \mathcal{F}$. Since $\mathcal{L}_s$ is finite, we can effectively compute the finite list $\{\mathcal{F}_i\}_{i \leq k}$. Let $\mathcal{L}_{s,0} = \mathcal{L}_s$. Given $\mathcal{L}_{s,i}$ defined, let $\mathcal{L}_{s,i+1}$ be the common extension of $\mathcal{L}_{s,i}$ and $\mathcal{F}_i$ as computed in (2). Finally, let $\mathcal{L}_{s+1} = \mathcal{L}_{s,k}$. Note that, we always add the smallest number which is not in $\mathcal{L}_{s,i}$ if we have to.

It can be verified every $(\mathcal{L}_s, \leq_{\mathcal{L}_s})$ is uniformly computable and so the structure $(\mathcal{L}, \leq_{\mathcal{L}})$ is computable.

To see $\mathcal{L}$ is universal. Fix a countable u.s.l. $\mathcal{A} = \{a_i\}_{i \in \mathbb{N}}$. We define partial embedding f_j of a sub u.s.l. of $\mathcal{A}$ into $\mathcal{L}$ inductively as follow. Let $f_0 = \emptyset$. Given f_j defined. Let i be the least such that $a_i \notin \operatorname{dom} f_j$, let s be the least such that $\operatorname{ran} f_j \subset \mathcal{L}_s$. By (1), there is a finite sub u.s.l. $\mathcal{F}$ of $\mathcal{A}$ generated from $\operatorname{dom} f_j \cup \{a_i\}$. Now by our construction, $\mathcal{F}$ is embedded into $\mathcal{L}_{s+1}$. Let f_{j+1} be the embedding. $\square$

Exercise 5.1.32. *Every countable usl $\mathcal{L}$ can be embedded in $\mathcal{D}$ and even in $\mathcal{D}(\leq 0')$ preserving $\vee$ as well as $\leq$ (and 0 if $\mathcal{L}$ has a least element).*

Proof. Let $(\mathbb{N}, \leq_{\mathcal{L}}, \vee_{\mathcal{L}})$ be the computable universal countable u.s.l. we build in the last proof. And $\{C_i\}_{i \in \mathbb{N}}$ is a sequence of very independence sets uniformly $\leq_T 0'$ as we build in Exercise 5.1.26.

For each $i \in \mathbb{N}$, we define $A_i = \left\{\langle j, x\rangle \mid i \nleq_{\mathcal{L}} j \text{ and } x \in C_j\right\}$. [2]

Each $A_i \leq_T 0'$ because $\leq_{\mathcal{L}}$ is a computable relation on $\mathbb{N} \times \mathbb{N}$ and $\left\{\langle j, x\rangle \mid x \in C_j\right\} \leq_T 0'$.

Now we show $i \mapsto [A_i]$ is really an embedding.

Assume $i_0 \leq_{\mathcal{L}} i_1$. Then $i_0 \nleq_{\mathcal{L}} j$ implies $i_1 \nleq_{\mathcal{L}} j$. To see if $\langle j, x\rangle \in A_{i_0}$: First, check if $i_0 \nleq_{\mathcal{L}} j$. If so, then $\langle j, x\rangle \notin A_{i_0}$; if not, $\langle j, x\rangle \in A_{i_0}$ if and only if $\langle j, x\rangle \in A_{i_1}$. Hence, $A_{i_0} \leq_T A_{i_1}$.

Assume $i_0 \nleq_{\mathcal{L}} i_1$. Then $A_{i_0}[i_1] = C_{i_1}$. But $A_{i_1}[i_1] = \emptyset$. Note that $A_{i_1} \leq_T \left\{\langle j, x\rangle \mid j \neq i_1 \text{ and } x \in C_j\right\}$ [again, because $\leq_{\mathcal{L}}$ is computable]. So if $C_{i_1} \leq_T A_{i_1}$, then $C_{i_1} \leq_T \left\{\langle j, x\rangle \mid j \neq i_1 \text{ and } x \in C_j\right\}$, violates $\{C_i\}_{i \in \mathbb{N}}$ being very independent. Therefore, $C_{i_1} \nleq_T A_{i_1}$. Again, since $C_{i_1} \leq_T A_{i_0}$, we have $A_{i_0} \nleq_T A_{i_1}$. This also shows that $i \mapsto [A_i]$ is one-to-one.

Fix i_0, i_1, we show $A_{i_0 \vee_{\mathcal{L}} i_1} \equiv_T A_{i_0} \oplus A_{i_1}$. Note that $A_{i_0 \vee_{\mathcal{L}} i_1} = \left\{\langle j, x\rangle \mid (i_0 \nleq_{\mathcal{L}} j \text{ or } i_1 \nleq_{\mathcal{L}} j) \text{ and } x \in C_j\right\}$. Clearly, $A_{i_0 \vee_{\mathcal{L}} i_1} \leq_T A_{i_0} \oplus A_{i_1}$. To see if $\langle j, x\rangle \in A_{i_0}$: Again, we first decide if $i_0 \nleq_{\mathcal{L}} j$. if so, we ask $A_{i_0 \vee_{\mathcal{L}} i_1}$; if not, return "no". $\square$

Exercise 5.2.5. *For every countable set of nonrecursive degrees there is a degree incomparable with each of them.*

Proof. Fix $\mathcal{C} = \{C_i\}_{i \in \mathbb{N}}$. We want to build A satisfying the following requirement.

$$\begin{aligned} &P_{i,e} \qquad C_i \neq \Phi_e^A, \\ &Q_{i,e} \qquad A \neq \Phi_e^{C_i}. \end{aligned}$$

We build $A = \bigcup_s \alpha_s$ as follow.

Let $A_0 = \emptyset$. At stage $s+1$:

When we act for $P_{i,e}$, we ask if $\exists x \exists \alpha \supset \alpha_s \Phi_e^\alpha(x) \downarrow \neq C_i(x)$. If so, let α_{s+1} be the least such α; otherwise, let $\alpha_{s+1} = \alpha_s$.

When we act for $Q_{i,e}$, we ask if $\Phi_e^{C_i}(|\alpha_s|) \uparrow$. If so, let $\alpha_{s+1} = \alpha_s$; otherwise, let $\alpha_{s+1} = \alpha_s^\frown\{1 - \Phi_e^{C_i}(|\alpha_s|)\}$.

The verification is routine. $\square$

[2] We need a specific definition of A_i than merely say $A_i = \oplus\{C_j \mid i \nleq_{\mathcal{L}} j\}$.

Exercise 5.2.6. *Every maximal antichain in $\mathcal{D}$ other than $\{0\}$ is uncountable.*

Proof. By Exercise 5.2.5, every countable antichain that is not $\{0\}$ is not maximal. □

Exercise 5.2.7. *Every maximal independent set of degrees other than $\{0\}$ is uncountable*

Proof. Similar with Exercise 5.2.6. □

Exercise 5.2.12. *For every $A >_T 0$ and $A \not\geq_T 0'$, there is B such that $A|_T B$ and $B' \leq_T A \oplus 0'$.*

This is a proof I heard from Ted Slaman. It is also the standard method to split $0'$. See also (Slaman and Steel, 1989) for a construction of a minimal pair B_0 and B_1 such that $0' \equiv_T B_0 \oplus B_1$.

Proof. We build B_0 and B_1 both $\leq_T A \oplus 0'$ and they are agree only at the slots we reserve to code $0'$. Then comparing B_0 and B_1 together will reveal the slots and so $0'$. We also build B_i $(i = 0, 1)$ to meet the following requirements.

$$P_{e,i}: \qquad \Phi_e^{B_i} = A \Rightarrow A \text{ is computable.}$$

First, we let $\beta_{0,0} = \beta_{1,0} = \emptyset$.

At stage $s + 1$ where $s = \langle e, i\rangle$, given $\beta_{0,s}$ and $\beta_{1,s}$ constructed, we act for $P_{e,i}$. We ask $0'$ if there is an e-split ahead of $\beta_{i,s}$, namely if

$$\exists x, \tau_0, \tau_1 \ \Phi_e^{\beta_{i,s}\frown\tau_0}(x) \downarrow \neq \Phi_e^{\beta_{i,s}\frown\tau_1}(x) \downarrow .$$

If exists, choose the least set of e-split $\langle x, \tau_0, \tau_1\rangle$, by consulting A, we choose $\tau = \tau_j$ such that $\Phi_e^{\beta_{i,s}\frown\tau_j}(x) \neq A(x)$. Let $\tilde{\tau}$ be the *complement* of τ, namely $|\tilde{\tau}| = |\tau|$ and $\tilde{\tau}(x) = 1 - \tau(x)$ for all $x \in |\tau|$.If there is no e-split ahead, we let $\tau = \tilde{\tau} = \emptyset$. Finally, we let $\beta_{i,s+1} = \beta_{i,s}\frown\tau\frown\{0'(s)\}$ and $\beta_{1-i,s+1} = \beta_{1-i,s}\frown\tilde{\tau}\frown\{0'(s)\}$. This completes the construction

Verification.

Note that the whole construction is in $A \oplus 0'$.

(1) To see $A \not\leq_T B_i$ for each $i = 0, 1$. Assume to contradiction that $A = \Phi_e^{B_i}$. Then there will be no e-split ahead of $\beta_{i,s}$ for $s = \langle e, i\rangle$. By enumerating $\sigma \supset \beta_{i,s}$, we will eventually compute $A(x) = \Phi_{e,|\sigma|}^{\sigma}(x)$ for all x.

(2) Apparently, $B_0 \oplus B_1 \geq_T 0'$. Therefore, A cannot compute both of them, i.e., there is $i = 0, 1$ such that $B_i \not\leq_T A$. Together with (1), there is B_i such that $B_i|_T A$.

(3) To see $B_i' \leq A \oplus 0'$ for each $i = 0, 1$. Fix e, to see if $e \in B_i'$, by s-m-n lemma, we find $f(e)$ (f recursive) such that $\Phi_{f(e)}^X(x) \simeq \Phi_e^X(e)$ for all $x \in \mathbb{N}$ and $X \subset \mathbb{N}$. Construct in $A \oplus 0'$, we will get $\beta_{i,s}$ where $s = \langle f(e), i\rangle$. We ask $0'$ if there is an $f(e)$-split ahead. If so, then $\Phi_e^{B_i}(e) \downarrow$ as $\Phi_{f(e)}^{B_i}(x) \downarrow$ for some x; If no, we ask $0'$ if $\Phi_{f(e)}^{\beta_{i,s}}(0) \downarrow$. □

Exercise 5.3.2. *Prove that all pairs of relations between A and B ($<_T, \leq_T, \equiv_T, |_T$) on the one hand and A' and B' on the other hand not prohibited by the known facts that $A <_T A'$ and $A \leq_T B \Rightarrow A' \leq_T B'$ are possible.*

Proof. We show $A|_T B$ and $A' <_T B'$ is possible.

Fix a set $A >_T 0$ being in a low degree, i.e. $A' \equiv 0'$. This can be given by the Kleene and Post Theorem. We would like to build a high B, namely $B' \equiv_T 0''$, such that $A \not\leq_T B$. Note that we will also have $B \not\leq_T A$, which follows from $B' \equiv_T 0'' \not\leq_T 0' \equiv_T A'$.

We define $B = \bigcup_s \beta_s$, let $\beta_0 = \emptyset$.

At stage $2s+1$, we ask if $\exists \beta \supset \beta_{2s} \Phi_s^\beta(s) \downarrow$. If so, we choose the least such β; if no, let $\beta = \beta_{2s}$. In either case, we let $\beta_{2s+1} = \beta^\frown\{0''(s)\}$.

At stage $2s+2$. We ask if $\exists x \exists \beta \supset \beta_{2s+1} \Phi_s^\beta(x) \downarrow \neq A(x)$. If so, let β_{2s+2} be the least such β. If not, let $\beta_{2s+2} = \beta_{2s+1}$.

Verification.

Note that $\langle \beta_s \rangle_s \leq A' \oplus 0''$. Since $A' \equiv_T 0'$, we have $\langle \beta_s \rangle_s \leq 0''$.

(1) To see $B' \leq_T 0''$. Given s, in $0''$, we can compute β_{2s}. Also in $0''$ (actually in $0'$), we can answer if $\exists \beta \supset \beta_{2s} \Phi_s^\beta(s) \downarrow$.

(2) To see $0'' \leq_T B'$. We show actually $0'' \leq_T B \oplus 0'$. We need to show that $t \mapsto |\beta_t|$ is recursive in $B \oplus 0'$. Clearly, $|\beta_0| = 0$. Given $|\beta_{2s}|$ and B, we know β_{2s} and we ask a $0'$ question to get $|\beta_{2s+1}|$. In the next step, we ask a $A'(\equiv_T 0')$ question to get $|\beta_{2s+2}|$. Now, $0''(s) = B(|\beta_{2s+1}| - 1)$.

(3) To see $A \not\leq_T B$. Assume $A = \Phi_e^B$. At stage $2e+2$, we would have $\forall x \forall \beta \supset \beta_{2e+1}(\Phi_e^\beta(x) \downarrow = A(x) \vee \Phi_e^\beta(x) \uparrow)$. Now we show A is recursive. To compute $A(x)$, we just search for a $\beta \supset \beta_{2e+1}$ and $s \in \mathbb{N}$ such that $\Phi_{e,s}^\beta(x) \downarrow$. We will find such β because $B \supset \beta_{2e+1}$ and $\Phi_e^B(x) \downarrow$. Now $A(x) = \Phi_{e,s}^\beta(x)$. □

Exercise 5.3.3 (Jump inversion preserving partial order)**.** *Prove that given any finite set $\mathcal{S}$ of Turing degrees $\geq_T \mathbf{0}'$, there is a set $\mathcal{T}$ of degrees such that $(\mathcal{T}, \leq_T)$ and $(\mathcal{S}, \leq_T)$ are isomorphic as partial orders and the isomorphism is given by the Turing jump operator.*

The original proof can be found in (Sacks, 1961).

Proof. Let $\mathcal{S} = \{S_k\}_{k<n}$. For each $k < n$, let $I_k = \{j < n \mid S_j \leq_T S_k\}$. We define a sequence of sets $\{A_k\}_{k<n}$ and let $B_k = \bigoplus_{j \in I_k} A_j$. We want

$$B_k' \equiv_T S_k$$

for every $k < n$. Clearly, if $S_i \leq_T S_j$, then $B_i \leq_T B_j$; and if $S_i \not\leq_T S_j$, then $B_i \not\leq_T B_j$.

We approximate $A_k = \bigcup_s \alpha_{k,s}$. Meanwhile, we also build $A_k^* = \bigcup_s \alpha_{k,s}^*$ leaving the slots for coding S_k blank. The intention is we would like to have $\langle \alpha_{k,s}^* \rangle_{k,s} \leq 0'$. Note that we always keep $|\alpha_{k,s}| = |\alpha_{k,s}^*|$. We enumerate following requirements.

$$\begin{array}{ll} P_x & \text{code } S_k(x) \text{ into } A_k \text{ for every } k < n, \\ R_{e,k} & \text{make sure } \Phi_e^{B_k}(e) \downarrow \text{ if possible.} \end{array}$$

Let $\alpha_{k,0} = \alpha_{k,0}^* = \emptyset$ for all $k < n$.

At stage $s+1$. Assume $\alpha_{k,s}$ and $\alpha_{k,s}^*$ are defined for all $k < n$.

If it is the stage we should act for P_x, we let $\alpha_{k,s+1} = \alpha_{k,s}{}^\frown\{S_k(x)\}$, and we let $\alpha_{k,s+1}^* = \alpha_{k,s}^*{}^\frown\{0\}$.

If it is the stage we should act for $R_{e,k}$. Assume we have already acted for P_x for every $x < m-1$. Then there are $|I_k| \cdot 2^m$ many ways to fill the slots in $\alpha_{j,s}^*$ for $j \in I_k$. Let $\{\alpha_{j,s}^p\}_{p<|I_k| \cdot 2^m}$ lists all the possibilities. By induction on $p < |I_k| \cdot 2^m$, we define μ_j^p for $j \in I_k$. Let $\mu_j^{-1} = \emptyset$ for every $j \in I_k$. Assume we have μ_j^{p-1} defined for every $j \in I_k$. We ask $0'$ if

$$\exists \langle \nu_j^p \rangle_{j \in I_k} \Phi_e^{\bigoplus_{j \in I_k} (\alpha_{j,s}^p \frown \mu_j^{p-1} \frown \nu_j^p)}(e) \downarrow .$$

If the answer is positive, take the least such $\langle \nu_j^p \rangle_{j \in I_k}$ and let $\mu_j^p = \mu_j^{p-1} \frown \nu_j^p$ for $j \in I_k$; if the answer is negative, let $\mu_j^p = \mu_j^{p-1}$ $(j \in I_k)$. Finally, let $\alpha_{j,s+1} = \alpha_{j,s} \frown \mu_j^{|I_k| \cdot 2^m - 1}$ and $\alpha^*_{j,s+1} = \alpha^*_{j,s} \frown \mu_j^{|I_k| \cdot 2^m - 1}$ for all $j \in I_k$.

Verification.

Note, the construction of $\langle \alpha^*_{k,s} \rangle_{k,s}$ is completely in $0'$.

(1) $B'_k \leq_T S_k$. To see if $e \in B'_k$. In $0'$, we compute $\langle \alpha^*_{j,s+1} \rangle_{j \in I_k}$ where s is the stage we acted for $R_{e,k}$. Now S_k (which computes every S_j for $j \in I_k$) can give us $\langle \alpha_{j,s+1} \rangle_{j \in I_k}$. Again, $0'$ will tell us if $\Phi_e^{\bigoplus_{j \in I_k} \alpha_{j,s+1}}(e) \downarrow$, which is also the answer for $e \in B'_k$. Therefore, $B'_k \leq_T 0' \oplus S_k \leq_T S_k$.

(2) $S_k \leq_T B'_k$. To compute $S_k(x)$, we use $0'$ to compute $\alpha^*_{k,s}$ where $s+1$ is the stage we acted for P_x. Then we ask B_k the value of $A_k(|\alpha^*_{k,s}|) = S_k(x)$. Therefore, $S_k \leq_T 0' \oplus B_k \leq_T B'_k$. □

Exercise 5.4.3. *There is an independent set of degrees of size continuum.*

Proof. As in the proof of Theorem 5.4.1, we build a perfect tree T such that for any n and any distinct $A_1, \dots A_n, B \in [T]$, $B \not\leq_T \oplus\{A_1, \dots, A_n\}$. So the requirements would be

$$R_e : \qquad \forall (A_1, \dots A_n, B) \in [T]^{<\omega} (B \neq A_1, \dots A_n \to \Phi_e^{\oplus\{A_1, \dots, A_n\}} \neq B)$$

We approximate $T = \bigcup_s T_s$. For each s, T_s is a finite tree whose every nonmaximal node has two incomparable extensions in T_s. Let $T_0 = \{\emptyset\}$. At the stage $s+1$, we have T_s defined and we act for requirement R_s.

Let $\sigma_0, \dots, \sigma_{n-1}$ list all the maximal nodes in T_s. And we list all the pairs $\langle \bar{i}, j \rangle$ such that $\bar{i} \in n^{<n}$, $j < n$ and j is not in $\bar{i}$ as $\{\langle \bar{i}_k, j_k \rangle \mid k < l\}$. For each $i < n$, let $\sigma_{i,0} = \sigma_i$. For $k < l$, assume $\sigma_{i,k}$ is defined for all $i < n$, we deal with the pair$\langle \bar{\sigma}_{\bar{i}_k,k}, \sigma_{j_k,k} \rangle$. We ask if there is $\bar{\sigma}$ such that each string in it extending the corresponding one in $\bar{\sigma}_{\bar{i}_k,k}$ respectively and $\Phi_s^{\oplus \bar{\sigma}}(|\sigma_{j_k,k}|) \downarrow$. If no such extensions, let $\sigma_{i,k+1} = \sigma_{i,k}$ for all $i < n$. Otherwise, let $\sigma_{j_k,k+1} = \sigma_{j_k,k} \frown \{1 - \Phi_s^{\oplus \bar{\sigma}}(|\sigma_{j_k,k}|)\}$, let $k+1$ extensions of strings in $\bar{\sigma}_{\bar{i}_k,k}$ be those correspondings in $\bar{\sigma}$ and keep other strings unchanged. Finally, we will have every $\sigma_{i,l}$ $(i < n)$ defined. Let T_{s+1} consists all $\sigma_{i,l} \frown \{0\}$, $\sigma_{i,l} \frown \{1\}$ for $i < n$ and their initial segments.

Verbification. $T = \bigcup_s T_s$ is clearly a perfect tree by construction. To see R_e is satisfied. Fix distinct $\bar{A}_{\bar{i}}$ and B in $[T]$. By padding lemma, there is always an s such that R_e and R_s are equivalent and the initial segments of $\bar{A}_{\bar{i}}$ and B has already been incomparable in T_s as $\langle \bar{\sigma}_{\bar{i}_k,k}, \sigma_{j_k,k} \rangle$. It is routine to check that $\Phi_s^{\oplus \bar{A}_{\bar{i}}} \neq B$. □

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