16. Relative ML random and 2-randomness

14:44

A natural way to get stronger notion of randomness is to have weaker notion of test, i.e. a random set should pass more tests

Recall that, a M-L test is a uniformly c.e. sequence

We can relax the definition by require it to be only uniformly c.e. relative in some oracle A

Recall, we say B is c.e. in A iff

$$B = W_e^A = \{n \mid \Phi_e^A(n)\downarrow\} \text{ for some } e.$$

$\{U_n\}_{n\in\omega}$ is uniformly c.e. in A iff there are basis $\{R_e\}_{e\in\omega}$ of $\{U_n\}_{n\in\omega}$ which is uniformly c.e.

Def Z is M-L random relative to A iff Z passes all M-L tests $\{U_n\}_{n\in\omega}$ uniformly c.e. in A

Similarly, we can define the relativised version of K

A prefix-free oracle machine is an oracle machine who domain is prefix-free

$K_M^A(\tau) = K_{M^A}(\tau)$ = the length of the shortest δ s.t. $M^A(\delta) = \tau$

where M is a prefix-free oracle machine, A is an oracle

$K^A(\tau) = K_U^A(\tau)$ for some universal prefix-free oracle machine

Exe there is an universal prefix-free oracle machine

Also we can define a (super) martingale $d: 2^{<\omega} \to \mathbb{R}^{\geq 0}$ are c.e. in A

Fact TFAE

1) Z is M-L random relative to A

2) There is a constant c such that for any n

$K(Z\upharpoonright n) \geq n - c$

3) There is no (super) martingale d c.e. in A wins on Z

Proof relativize

Fact If $Z \leq_T A$. Then Z is not M-L random relative to A

Proof Assume $Z = \Phi^A$.

Define $U_n = [\Phi^A \upharpoonright_n]$ which is a ML test relative to A $\square$

Theorem Let $A, B \subseteq \mathbb{N}$. Then

$A \oplus B$ is M-L random $\Leftrightarrow$ B is M-L random and A is M-L random relative to B

Proof See section 6.9

Corollary if A, B are M-L random, then

A is M-L random relative to B $\Leftrightarrow$ B is M-L random relative to A

Corollary if $A \oplus B$ is M-L random, then $A |_T B$

Definition We say a set $A \subseteq \mathbb{N}$ is low if $A' \equiv_T \phi'$

Now, since we always have $\phi' \leq_T A'$, so it is equivalent to say A is low if $A' \leq_T \phi'$

Recall a set A is Δ_2^0 iff $A \leq_T \phi'$

since we always have $A \leq_T A'$, so a low set is always Δ_2^0

And is A is low, then $A <_T \phi'$

Thm There is a simple low set.

Corollary There is a non-computable low set

Def We say a set A is general low if $A' \equiv_T A \oplus \phi'$

Fact If A is Δ^0_2 and generalized low, then A is low.

But generalized low set may not be Δ^0_2

Thm If Z is Δ^0_2 and Z is M-L random relative in A

Then A is generaliz low

Proof Z is Δ^0_2 so there is a function $f: \mathbb{N} \to \mathbb{N}$ s.t.

For each n, $f(n) = \mu s\ \forall t \geq s\ (Z_s \upharpoonright n = Z_t \upharpoonright n)$

Note $f \leq_T \phi'$

We want to compute $A' = \{ e \mid \Phi^A_e(e)\downarrow \}$

Define

$\sigma_e = Z_{s_e} \upharpoonright e+1$ if s_e is the first stage that $\Phi^A_{e,s_e}(e)\downarrow$

if no such s_e, it is undefined

Let $U_n = [\![\{\sigma_e \mid e \geq n\}]\!]$

Then $\{U_n\}_{n\in\mathbb{N}}$ is uniformly c.e. in A

And $\mu U_n \leq \sum_{e\geq n} \sigma_e \leq \sum_{e\geq n} 2^{-(e+1)} = 2^{-n}$

so $\{U_n\}_{n\in\omega}$ is a M-L test relative to A

$-$ $\cap_n U_n$, i.e. $Z \in [\![\sigma_e]\!]$ only for finitely many e.

i.e. $Z_{s_e} \upharpoonright e+1 \not< Z$ only for almost all e

i.e. $s_e \leq f(e)$ for almost all e

So for almost all e, to decide if $\Phi^A_e(e)\downarrow$,

we need only to check if $\Phi^A_{e,f(e)}(e)\downarrow$ which is computable in

$A \oplus f$ and so $A \oplus \phi'$ $\square$

There is a canonical hierarchy of stronger notion of randomness

Def A set A is n-random if A is M-L random relative to $\phi^{(n-1)}$

Recall A is 1-random iff A is M-L random

We look at 2-randomness (M-L random relative to $\emptyset'$)

It seems that 2-randomness is already fixing the drawbacks (too weak) of M-L randomness

Fact Ω is not 2-randomness

Proof $\Omega \equiv_T \emptyset'$

Let $U_n = [\![\Omega \upharpoonright n]\!]$ $\square$

Exe No left c.e. set is 2-random

We show 2-random sets are powerless as oracle

Thm If A is 2-random, then A is generalized low

Proof Assume A is 2-random then A is M-L random relative to $\emptyset'$ and so to Ω. Since Ω is M-L random so $A \oplus \Omega$ is M-L random, therefore

Ω is M-L random relative to A

Since Ω is Δ^0_2 so A is generalized low $\square$