15. Martin-Lof randomness is too weak

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The notion "Martin-Löf randomness" is too weak in the sense that the halting probability Ω is Martin-Löf random, but it is left-c.e. and Turing equivalent to $\emptyset'$, therefore it computes every Δ^0_2 set.

Kŭcera – Gács Theorem says Martin-Löf random sets can be arbitrary powerful as oracles.

<u>Theorem (Kŭcera – Gács)</u>

For every set A there is a ML-random set Z s.t. $A \leq_T Z$

<u>Lemma</u> Fix measurable $S \subseteq 2^\omega$, $\delta \in 2^{<\omega}$ and $r \in \mathbb{N}$

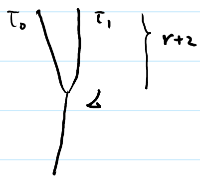
If $\mu(S \cap [\![\delta]\!]) \geq 2^{-(r+1)} \cdot 2^{-|\delta|}$

Then there are distinct $\tau_0, \tau_1 \succ \delta$ s.t. $|\tau_0| = |\tau_1| = |\delta| + r + 2$

and $\mu(S \cap [\![\tau_i]\!]) > 2^{-(r+2)} \cdot 2^{-|\tau_i|}$ for $i = 0, 1$

<u>Proof</u> Let $C = \{\tau \mid |\tau| = |\delta| + r + 2 \text{ and } \tau \succ \delta\}$

Note $|C| = 2^{r+2}$



Then $\mu(S \cap [\![\delta]\!]) = \sum_{\tau \in C} \mu(S \cap [\![\tau]\!])$ (by measurability)

and it must be that $\mu(S \cap [\![\tau]\!]) \leq \mu[\![\tau]\!] = 2^{-|\tau|} = 2^{-|\delta| - (r+2)}$

Assume $\mu(S \cap [\![\tau_0]\!]) = 2^{-|\delta| - (r+2)}$ for some $\tau_0 \in C$ and $\mu(S \cap [\![\tau]\!]) \leq 2^{-(r+2)} \cdot 2^{-|\tau|}$
$= 2^{-|\delta| - 2(r+2)}$

Then $\mu(S \cap [\![\delta]\!]) \leq 2^{-|\delta| - (r+2)} + (2^{r+2} - 1) \cdot 2^{-|\delta| - 2(r+2)}$

$= 2^{-|\delta| - (r+1)} - 2^{-|\delta| - 2(r+2)}$

$< 2^{-|\delta| - (r+1)}$ $\rightarrow\leftarrow$ $\square$

<u>Proof of the theorem</u>

Let $\{U_n\}_{n \in \omega}$ be a universal ML-test, then U_1 is a c.e. class, $\mu(U_1) \leq 2^{-1}$

and so $\mu(2^\omega - U_1) \geq 1/2$, moreover sets in $2^\omega - U$ are all M-L random sets.

and so $\mu(2^\omega - \mathcal{U}_1) \geq 1/2$, moreover sets in $2^\omega - \mathcal{U}$ are all M-L random sets.

Let $S = 2^\omega - \mathcal{U}_1$, we use sets in S to code arbitrary sets $A \in 2^\omega$

Let $\delta_{\langle\rangle} = \langle\rangle$. then $\mu(S \cap [\![\delta_{\langle\rangle}]\!]) = \mu(S) \geq 2^{-1}$

Given δ_τ defined. Let $r = |\tau|$. IH: $\mu(S \cap [\![\delta_\tau]\!]) \geq 2^{-(|\tau|+1)}$
$= 2^{-(r+1)} \cdot 2^{-|\delta_\tau|}$

By the previous Lemma, there are distinct $\delta_0, \delta_1 \succ \delta_\tau$ s.t.

$|\delta_0| = |\delta_1| = |\delta_\tau| + r + 2$ and

$\mu(S \cap [\![\delta_i]\!]) > 2^{-(r+2)} \cdot 2^{-|\delta_i|}$

Let $\delta_{\tau 0}, \delta_{\tau 1}$ be the leftmost and rightmost such strings, then

$\mu(S \cap [\![\delta_{\tau i}]\!]) > 2^{-(|\tau|+2)} \cdot 2^{-|\delta_{\tau i}|}$
$= 2^{-(|\tau i|+1)} \cdot 2^{-|\delta_{\tau i}|}$

Now we have defined δ_τ for any $\tau \in 2^{<\omega}$

Given $A \in 2^\omega$, let $Z = \bigcup_{\tau < A} \delta_\tau$. clearly $Z \in S$, so Z is M-L random,

Claim $A \leq_T Z$

To extract A from Z, Assume $A \upharpoonright n = \tau$ is computed

To decide $A(n)$:

Note that for every τ, $|\delta_\tau| = |\tau| \cdot (|\tau| + 3)/2$

$[\, |\delta_{\langle\rangle}| = f(0) = 0,\ |\delta_{\tau i}| = f(r+1) = |\delta_\tau| + |\tau| + 2 = f(r) + r + 2 \,]$

So we know $Z \upharpoonright f(n) = \delta_\tau$, we want to decide whether

$Z \upharpoonright f(n+1) = \delta_{\tau 0}$ or $= \delta_{\tau 1}$ i.e.

if $Z \upharpoonright f(n+1)$ is the leftmost or the right most δ s.t.

$\delta \succ \delta_\tau = Z \upharpoonright f(n)$ and $\mu(S \cap [\![\delta]\!]) \geq 2^{-(n+2)} \cdot 2^{-f(n+1)}$

Key: $\mathcal{U}_1 = 2^\omega - S$ is c.e. (or Σ^0_1)

So we can enumerate $\{\rho \mid [\![\rho]\!] \subset \mathcal{U}_1\}$, and if for any $\theta \succ \delta_\tau$ $|\theta| = f(n+1)$

So we can enumerate $\{\rho \mid [\![\rho]\!] \subset \mathcal{U}_1\}$, and if for any $\theta \succ \sigma_\tau$ $|\theta| = f(n+1)$

if $\mu(S \cap [\![\theta]\!]) < 2^{-(n+2)} \cdot 2^{-f(n+1)}$ i.e. $\mu(\mathcal{U} \cap [\![\theta]\!]) \geq 2^{-(n+2)} \cdot 2^{-f(n+1)}$

we will find it in finitely many steps.

$= Z \upharpoonright f(n+1)$

If σ is the leftmost, in finitely many steps, we will find all θ to the left of σ s.t. $\mu(S \cap [\![\theta]\!]) < 2^{-(n+2)} \cdot 2^{-f(n+1)}$

$= Z \upharpoonright f(n+1)$

If σ is the rightmost, in - - - - -

right - - - - - - .

Therefore, we can decide if σ is σ_{τ_0} or σ_{τ_1} in finitely many steps

in either case we know $A(n) = 0$ or 1 $\square$