14. No optimal c.e. martingale

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Recall that, if $d = c \cdot f$, then $S[d] = S[f]$

<u>Def</u> A c.e. (super) martingale d is optimal if for each (super) martingale f, there is a constant c such that $c \cdot d(\sigma) \geq f(\sigma)$ for all σ.

<u>Fact</u> An optimal c.e. (super) martingale d is an universal c.e. (super) martingale

i.e. for each c.e. (super) martingale f, $S[f] \subseteq S[d]$

<u>Proof</u> Assume $c \cdot d(\sigma) \geq f(\sigma)$ for all σ

$A \in S[f]$, then for each $m > 0$, there is $n \in \mathbb{N}$ s.t.

$f(A \upharpoonright n) \geq c \cdot m$, and so $d(A \upharpoonright n) \geq m$, so $A \in S[d]$ □

<u>Theorem</u> There is an optimal c.e. supermartingale

<u>Proof</u> We can enumerate all supermartingale (module multiplication) as follow

Actually, we enumerate all supermartingale d s.t. $d(\langle\rangle) \leq 1$

(And so $d(\sigma) \leq 2^{|\sigma|}$ for all σ)

Recall that $f: 2^{<\omega} \to \mathbb{R}^{\geq 0}$ is c.e. if $\{f(\sigma) \mid \sigma \in 2^{<\omega}\}$ can be uniformly approximate from left (uniformly left c.e.) which is witness by, say, Φ_e.

effectively define $\Phi_{p(e)}$ be a modified program: once

$f(\sigma)$ tend to increase and threatens to violates

$f(\sigma) \leq 2^{|\sigma|}$ or $f(\sigma i)$ tend to increase and threatens to violates $\frac{f(\sigma 0) + f(\sigma 1)}{2} \leq f(\sigma)$.

Stop it!

Let $f_e: 2^{<\omega} \to \mathbb{R}^{\geq 0}$ be the c.e. function witness by $\Phi_{p(e)}$

Then $\{f_e\}_{e \in \omega}$ is an effective enumeration of all c.e. supermartingale

Let $d: 2^{<\omega} \to \mathbb{R}^{\geq 0}$ be s.t.

Then $\{t_e\}_{e\in\omega}$ is an effective enumeration of all c.e. supermartingale

Let $d: 2^{<\omega} \to \mathbb{R}^{\geq 0}$ be s.t.

$$d(\sigma) = \sum_{e=0}^{\infty} 2^{-n} t_e(\sigma)$$

Then d is a c.e. supermartingale and d is optimal $\square$

Note $\Phi_e \to \Phi_{p(e)}$ can only ensure the corresponding $f: 2^{<\omega} \to \mathbb{R}^{\geq 0}$ is supermartingale, i.e. if f is a martingale witness by Φ_e, it is fairly possible that f witness by $\Phi_{p(e)}$ is a supermartingale but not a martingale.

<u>Theorem</u> There is no computable enumeration of all c.e. martingale

<u>Proof</u> Assume to contradiction that $d_0, d_1 \cdots$ is a computable enumeration of all c.e. martingale

From this, we produce an computable enumeration $f_0, f_1 \cdots$ of all c.e. martingales that are not constantly zero functions as follow:

we uniformly approximate each $d_i(\langle\rangle)$ from left,

If d_{i_n} is the nth we found which is not constantly zero,

then let $f_i = d_{i_n}$

By a diagonal argument, we produce a strictly positive c.e. martingale d not equal to any f_i

For $\sigma \neq \langle\rangle$, define $\sigma^- = \sigma \upharpoonright (|\sigma|-1)$. (delete the last digital)

$\sigma^c = \sigma^- (1-\sigma(|\sigma|))$ (flipping the last digital)

once $f_0(\langle\rangle)[s] > 0$, let $d(\langle\rangle) = \frac{f_0(\langle\rangle)[s]}{2}$

Assume $d\upharpoonright n+1$ is defined,

search for s and $\sigma \in 2^{n+1}$ s.t. $f_{n+1}(\sigma)[s] > 0$

[Note since f_{n+1} is a martingale (not only super), and $f_{n+1}(\langle\rangle) > 0$,

so $\sum_{\tau \in 2^n} f(\tau) = 2^n \cdot f(\langle\rangle) \neq 0$, thus there must be some $\sigma \in 2^n$ s.t. $f(\sigma) > 0$]

Let $d(\sigma) = \min\left\{ d(\sigma^-), \frac{f_{n+1}(\sigma)[s]}{2} \right\}$

and $d(\sigma^c) = 2d(\sigma^-) - d(\sigma)$

And for all other $\tau \in 2^{n+1}$, let $d(\tau) = d(\tau^-)$

Clearly d is c.e. (even computable) and d is a martingale

furthermore for each n, there is $\delta \in 2^n$ s.t.

$$d(\delta) \le \frac{f(\delta)[s]}{2} < f(\delta)$$

for some s

□

Theorem There is no optimal c.e. martingale

Proof Fix a c.e. martingale d. we show d is not optimal

i.e. we build a c.e. martingale f s.t. for each c, there is δ with $f(\delta) \ge c \cdot d(\delta)$

We actually build a uniform sequence of c.e. martingale

$$f_0, f_1, \dots$$

s.t. $f_n(\langle\rangle) = 1$ and $f_n(\delta) \ge 2^{2n} d(\delta)$ for some δ

If it is achieved, Let $f = \sum_n 2^{-n} f_n$.

Then f is a c.e. martingale and for each n, there is δ s.t.

$$f(\delta) \ge 2^{-n} \cdot f_n(\delta) \ge 2^n d(\delta)$$, we are done!

To build the sequence:

Given n, Let $f_n(\langle\rangle) = 1$

Stage 0 Let $\delta_0 = \langle\rangle$

We approximate $d(\langle\rangle)$ from left by $d(\langle\rangle)[s]$

If $d(\langle\rangle)[s] < 2^{-2n}$, let $f_n(0^{s+1}) = 2^{s+1}$ and $f_n(\tau) = 0$ for all other $\tau \in 2^{s+1}$.

Once $d(\langle\rangle)[s_0] \ge 2^{-2n}$, we wait until t_0 s.t.

$$\sum_{\tau \in 2^{s_0+1}} d(\tau)[t_0] \ge 2^{s_0+1-2n}$$

[Note $\sum_{\tau \in 2^{s_0+1}} d(\tau) = d(\langle\rangle) \cdot 2^{s_0+1}$, so such t_0 must exists]
since d is a martingale

Let $i \in \{0,1\}$ be s.t. $d(0^{s_0} i)[t_0] \le d(0^{s_0}(1-i))[t_0]$ (choose the one d put less on it)

Let $i \in \{0,1\}$ be s.t. $d(0^{s_0} i)[t_0] \le d(0^{s_0}(1-i))[t_0]$

(choose the one d put less on it
then let f_n put all on it)

let $\delta_1 = 0^{s_0} i$.

Let $f_n(\delta_1) = 2^{s_0+1}$ and let $f_n(\tau) = 0$ for other $\tau \in 2^{s_0+1}$

Then $$\underbrace{\sum_{\tau \in 2^{s_0+1}} d(\tau)[t_0]}_{\ge 2^{s_0+1-2n}} - d(\delta_1)[t_0] \ge 2^{s_0-2n}$$

(since $d(\delta_1)[t_0] \le d(\delta_1^c)[t_0]$)

Put in mind that for any m,

if $d(\delta_1) \ge m \cdot 2^{-2n} f_n(\delta_1) = m \cdot 2^{s_0+1-2n}$,

then $$d(\delta_0) = d(\langle\rangle) = 2^{-(s_0+1)} \cdot \sum_{\tau \in 2^{s_0+1}} d(\tau)$$
$$\ge 2^{-(s_0+1)} (2^{s_0-2n} + m \cdot 2^{s_0+1-2n})$$
$$= (m+\tfrac{1}{2}) 2^{-2n}$$

<u>Stage $k+1$</u> start from δ_k

Approximate $d(\delta_k)$ by $d(\delta_k)[s]$

As long as $d(\delta_k)[s] < 2^{-2n} \cdot f_n(\delta_k)$

define $f_n(\delta_k 0^{s+1}) = 2^{|\delta_k|+s+1}$ and $f_n(\tau) = 0$ for all other $\tau \in 2^{|\delta_k|+s+1}$

Once $d(\delta_k)[s_k] \ge 2^{-2n} f_n(\delta_k) = 2^{|\delta_k|-2n}$

wait until t_k s.t. $\sum_{\tau \in 2^{s_k+1}} d(\delta_k \tau)[t_k] \ge 2^{|\delta_k|+s_k+1-2n}$,

which must occur.

Let $i \in \{0,1\}$ be s.t. $d(\delta_k 0^{s_k} i)[t_k] \le d(\delta_k 0^{s_k}(1-i))[t_k]$

Let $\delta_{n+1} = \delta_n 0^{s_n} i$, let $f_n(\delta_{k+1}) = 2^{|\delta_k|+s_k+1}$

$f_n(\tau) = 0$ for all other $\tau \in 2^{|\delta_k|+s_k+1}$

Similarly
$$\sum_{\tau \in 2^{s_k+1}} d(\delta_k \tau)[t_k] - d(\delta_{k+1})[t_k] \ge 2^{|\delta_k|+s_k-2n}$$

So for each m.

So For each m.

if $d(\sigma_{k+1}) \geq m \cdot 2^{-2n} t_n(\sigma_{k+1}) = m \cdot 2^{|\sigma_k| + s_k + 1 - 2n}$

then $d(\sigma_k) = 2^{-(s_k+1)} \sum_{\tau \in 2^{s_k+1}} d(\sigma_k \tau)$

$$\geq 2^{-(s_k+1)} \left(2^{|\sigma_k| + s_k - 2n} + m \cdot 2^{|\sigma_k| + s_k + 1 - 2n} \right)$$

$$= (m + \tfrac{1}{2}) \, 2^{|\sigma_k| - 2n}$$

$$= (m + \tfrac{1}{2}) \, 2^{s_k + 1 - 2n}$$

so $d(\langle\rangle) \geq (m + \tfrac{1}{2}) \, 2^{-2n}$

It cannot be the case that each σ_i is defined.

For otherwise, every time, we have $d(\sigma_{i-1}) \geq 2^{-2n} t_n(\sigma_{i-1})$

and so $d(\langle\rangle) \geq \frac{i+1}{2} 2^{-2n}$ for all i, which is impossible $\square$