13. Martingales

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The third approach to randomness is via unpredictability

Intuitively, unpredictability means if we gamble on what will happened, there is no fixed strategy can ensure either player will win eventually

To characterize this we introduce the notion: Martingale

Def A function $d: 2^{<\omega} \to \mathbb{R}^{\geq 0}$ ($= [0, +\infty)$) is a martingale if for all σ

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2}$$

It is a supermartingale if

$$d(\sigma) \geq \frac{d(\sigma 0) + d(\sigma 1)}{2}$$

A (super) martingale d succeeds on a set A if

$$\limsup_n d(A \upharpoonright n) = \infty$$

Define the succeed set of d

$$S[d] = \{A \in 2^\omega \mid d \text{ succeeds on } A\}$$

Idea: A martingale d is a betting strategy

Normally $d(\langle\rangle) > 0$ (empty sequence) (o.w. d is a constant function with value 0) denotes the starting capital

Note the exact value of $d(\langle\rangle)$ is actually irrelevant. (see below)

A player bets on a sequence of tosses of a coin, if it comes up heads or comes up tails.

If he win a bet (right on head / tail) he wins double of his stake, if not he losses the stake

At the first run, following the strategy, he bet

$d(0)/2$ on head

$d(1)/2$ on tail

Note $d(0)/2 + d(1)/2 = \frac{d(0)+d(1)}{2} = d(\langle\rangle)$ (if d is martingale)

$\leq$ (if d is supermartingale)

As god, we know he is playing on A, assume $A(0) = 0$

then the player will have $2 \cdot (d(0)/2) = d(0)$ at the end of this run.

Inductively, at the beginning of step n, the player will have capital $d(A\restriction n)$, and he follows the strategy bets $d(A\restriction n^\frown 0)/2$ on 0, $d(A\restriction n^\frown 1)/2$ on 1 for this run.

And he will have $d(A\restriction(n+1))$ at the end of this run.

Supermartingale allow the player to discard part of his capital (for better life)

<u>Fact</u> Let d be (super) martingale, $r \in \mathbb{R}^+$ is a positive real, and $f: 2^{<\omega} \to \mathbb{R}^{\geq 0}$ s.t. $f = r \cdot d$, i.e. $f(\sigma) = r \cdot d(\sigma)$ for all $\sigma \in 2^{<\omega}$

Then f is a (super) martingale. And $S[d] = S[f]$

<u>Fact</u> If $d_1, d_2 \cdots$ are (super) martingale such that $\Sigma_n d_n(\langle\rangle) < \infty$

then $\Sigma_n d_n$ (the function d s.t. $d(\sigma) = \Sigma_n d_n(\sigma)$) is a (super) martingale

<u>Kolmogorov's Inequality</u>

<u>Thm</u> Let d be a (super) martingale

(i) For any string $\sigma \in 2^{<\omega}$ and any prefix-free set S of extensions of σ, we have

$$\sum_{\tau \in S} 2^{-|\tau|} d(\tau) \leq 2^{-|\sigma|} d(\sigma)$$

(ii) Let $R_k = \{\sigma \mid d(\sigma) \geq k\}$. Then $\mu[\![R_k]\!] \leq \frac{d(\langle\rangle)}{k}$

Proof (i) First, note for $i = 0, 1$

$$\frac{d(\sigma^\frown i)}{2} \le \frac{d(\sigma^\frown i) + d(\sigma^\frown(1-i))}{2} \le d(\sigma)$$

By induction on $|\tau| - |\sigma|$, we can prove that if $\sigma \preceq \tau$, then

$$2^{-|\tau|+|\sigma|} \cdot d(\tau) \le d(\sigma) \qquad \text{so}$$

$$2^{-|\tau|} d(\tau) \le 2^{-|\sigma|} d(\sigma)$$

This is the case for $|S| = 1$

We prove the the cases for $|S| = n$ by induction

Assume $|S| = n+1$

Let $\nu \succeq \sigma$ be the longest s.t. every $\tau \in S$ extends ν.

Then $\{\tau \in 2^{<\omega} \mid \tau \succeq \nu 0\} \cap S = S_0 \ne \emptyset$
$\{\tau \in 2^{<\omega} \mid \tau \succeq \nu 1\} \cap S = S_1 \ne \emptyset$

For otherwise either $\nu 0$ or $\nu 1$ is longer than ν satisfying the same property.

So $1 \le |S_0| \le n$ and $1 \le |S_1| \le n$

By induction hypothesis,

$$\sum_{\tau \in S_0} 2^{-|\tau|} \cdot d(\tau) \le 2^{-|\nu|-1} \cdot d(\nu 0)$$

$$\sum_{\tau \in S_1} 2^{-|\tau|} \cdot d(\tau) \le 2^{-|\nu|-1} \cdot d(\nu 1)$$

So $\sum_{\tau \in S} 2^{-|\tau|} d(\tau) \le 2^{-|\nu|-1} \cdot (d(\nu 0) + d(\nu 1))$

$\le 2^{-|\nu|} d(\nu)$ [since $\frac{d(\nu 0) + d(\nu 1)}{2} \le d(\nu)$, by definition]

$\le 2^{-|\sigma|} d(\sigma)$ [the case $|S| = 1$]

Note, if $S = \{\tau_1, \tau_2, \cdots\}$ is infinite and $\sum_{i=0}^{\infty} 2^{-|\tau_i|} d(\tau_i) > 2^{-|\sigma|} d(\sigma)$

then there is already some n s.t. $\sum_{i=0}^{n} 2^{-|\tau_i|} d(\tau_i) > 2^{-|\sigma|} d(\sigma)$

So it is sufficient to prove the case when S is finite

(ii) Let $P \subseteq R_k$ be prefix-free s.t. $[\![P]\!] = [\![R_k]\!]$

Then $\mu[\![P]\!] = \sum_{\tau \in P} 2^{-\tau} \le \sum_{\tau \in P} 2^{-\tau} \cdot \frac{d(\tau)}{k}$ (since $\frac{d(\tau)}{k} \ge 1$)

$\le \frac{d(\epsilon)}{k}$ □

We say a (super) martingale is computably enumerable (c.e.) if its values are uniformly left c.e. i.e. there is a computable function $p: 2^{<\omega} \to \mathbb{N}$, given $\sigma \in 2^{<\omega}$, $W_{p(\sigma)}$ is a c.e. set of all "left" rational number $<_L d(\sigma)$

We say a (super) martingale is computable if its values are uniformly computable. i.e. there is a computable function $p: 2^{<\omega} \to \mathbb{N}$ given $\sigma \in 2^{<\omega}$, $\Phi_{p(\sigma)}$ is a program which decides if a rational number $q <_L d(\sigma)$ or not

Thm (Schnorr)

A set is 1-random iff there is no c.e. (super) martingale succeeds on it

Proof ($\Rightarrow$) Let $A \in 2^\omega$, d is a c.e. (super) martingale, s.t. $d(\epsilon) = 1$

Assume $A \in S[d]$. We define a M-L test to "capture" A.

Let $U_n = [\![\{\sigma \in 2^{<\omega} \mid d(\sigma) \geq 2^n\}]\!]$

Then $\mu U_n \leq \frac{d(\epsilon)}{2^n} = 2^{-n}$

Since d is c.e. $\{U_n\}_{n\in\omega}$ is uniformly c.e.

[Once we find $d(\sigma)$ exceeds 2^n, we put σ into the basis of U_n]

Therefore $\{U_n\}_{n\in\omega}$ is a M-L test.

Since $A \in S[d]$. For each n, there is m s.t. $d(A\upharpoonright m) \geq 2^n$
i.e. $A \in U_n$ so $A \in \bigcap_n U_n$. "captured" by the M-L test

So A is not 1-random.

($\Leftarrow$) If A is capture the M-L test $\{U_n\}_{n\in\omega}$.

Let $\{R_n\}_{n\in\omega}$ be a uniformly c.e. prefix free basices of $\{U_n\}_{n\in\omega}$

We define a sequence of martingale d_n bets on the fact that A will enter U_n

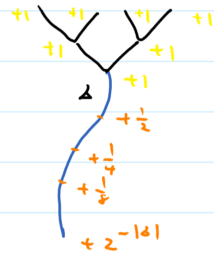
At beginning, we let every d_n be constant function with value 0, We uniformly enumerates $\{R_n\}_{n\in\omega}$, once σ enters R_k

We put more weight on betting σ as follow:

we add 1 to the value of $d_n(\tau)$ for all $\tau \succeq \sigma$

and we add $2^{k-|\sigma|}$ to $d_n(\sigma \upharpoonright k)$ for all $k < |\sigma|$.

In this way, we keep d_n as a martingale



Let $d = \sum_n d_n$

Then $d(\langle\rangle) = \sum_n \sum_{\sigma \in R_n} 2^{-|\sigma|}$

$= \sum_n \mu[R_n] = \sum_n \mu U_n \le \sum_n 2^{-n} < \infty$

So d is a martingale. clearly, d is c.e. by construction.

If $A \in \bigcap_n U_n$, then for each n, there is k. for all $i < n$.

$A \upharpoonright k \succeq \sigma_i$ for some $\sigma_i \in R_i$

then $d(A \upharpoonright k) \ge n$

so $A \in S[d]$ □

<u>Corollary</u> There is a universal c.e. martingale d, that is

for any c.e. martingale f, $S[f] \subseteq S[d]$

<u>Pf</u> Let $\{U_n\}_{n\in\omega}$ be a universal M-L test.

Use the argument in the previous theorem, define a martingale d s.t.

If $A \in \bigcap_n U_n$ then $A \in S[d]$

Now for each c.e. martingale f, if $A \in S[f]$, then A is not M-L random

so A is captured by the universal M-L test and so $A \in S[d]$ □