12. Universal Martin-Lof test and Solovay test

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Def A Martin-Löf test $\{U_n\}_{n\in\omega}$ is universal if for any Martin-Löf test $\{V_n\}_{n\in\omega}$, $\bigcap_n V_n \subseteq \bigcap_n U_n$

Therefore if a set A pass a universal Martin-Löf test, it passes all Martin-Löf test, so it is Martin Löf-random.

Theorem There is a universal Martin-Löf test

Proof We show that there is a effective enumeration of all M-L test.

Let $\{R^0_n\}_{n\in\omega}$, $\{R^1_n\}_{n\in\omega}$ ... be an effective enumeration of all uniformly c.e. subsets of $2^{<\omega}$

(we just enumerate the programs)

For each program P_e enumerates $\{R^e_n\}_{n\in\omega}$ (note, P_e actually enumerates $\{(n,i) \mid i\in R^e_n\}$)

effectively change it to be Q_e: First enumerates R^e_n uniformly as P_e do, once (n,i) try to enter R^e_n, $[\![R^e_n]\!]$ threatens to exceed 2^{-n} stop it.

Let $\{S^e_n\}_{n\in\omega}$ be the uniformly c.e. sets enumerated by Q_e

Then $\{S^0_n\}_{n\in\omega}$, $\{S^1_n\}_{n\in\omega}$ ... is an effective enumeration of all uniformly c.e. sets and $\mu[\![S^e_n]\!] \le 2^{-n}$

$$
\begin{array}{ccc}
S^0_0 & S^0_1 & S^0_2 \\
S^1_0 & S^1_1 & S^1_2 \\
S^2_0 & S^2_1 & S^2_2
\end{array}
\quad \cdots
$$

Let $U_n = \bigcup_e [\![S^e_{n+e+1}]\!]$

Then $\{U_n\}_{n\in\omega}$ is uniformly c.e.

And $\mu U_n = \sum_e \mu[\![S^e_{n+e+1}]\!]$

$$\le \sum_e 2^{-(n+e+1)} = 2^{-n}$$

Therefore $\{U_n\}_{n\in\omega}$ is a M-L test, and is universal. $\square$

Note, the ML test we built:

$$U_k = \{X \mid \exists n\, K(X\upharpoonright n) \le n-k\}$$ is already universal.

<u>Corollary</u> The class of 1-random sets has measure 1

<u>Def</u> A Solovay test is a sequence $\{S_n\}_{n\in\omega}$ of uniformly c.e. classes (of nonempty ones) such that $\sum_n \mu(S_n) < \infty$. A set A is solovay random if for every solovay test, A is in only finitely many S_n

<u>Theorem</u> A set is solovay random iff it is 1-random

<u>Proof</u>

($\Rightarrow$) Note every M-L test is a Solovay test.

So if a set A passes all Solovay tests, it passes all M-L tests

($\Leftarrow$) Suppose A is M-L random and $\{S_n\}_{n\in\omega}$ is a Solovay test,

We show A passes the test, namely, A is not contained in infinitely many S_n.

Note there is m s.t. $\sum_{n>m} \mu(S_n) < 1$

Let $U_k = [\![\{\sigma \mid \text{there are} \ge 2^k \text{ many } n>m \text{ s.t. } [\![\sigma]\!] \in S_n\}]\!]$

Then $\{U_k\}_{k\in\omega}$ is uniformly c.e.

And $\mu U_k \le 2^{-k} \sum_{n>m} \mu(S_n) < 2^{-k}$

every $[\![\sigma]\!] \subseteq U$ has been counted at least 2^k many times in $\sum_{n>m} \mu(S_n)$

So $\{U_k\}_{k\in\omega}$ is a M-L test

So $A \notin U_k$ for some k so A is in at most $2^k + m$ many S_n $\square$