11. Martin-Lof randomness

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Recall a sequence $A \in 2^\omega$ is 1-random iff there is $d \in \mathbb{N}$ s.t.

$$K(A\upharpoonright n) \geq n - d \quad \text{for all } n \in \mathbb{N}$$

We defined a specific 1-random real: Ω the halting probability

We showed $\Omega \equiv_T \emptyset'$ and Ω is 1-random

We will see almost all reals are 1-random. For now:

Proposition Let $A \in 2^\omega$. If there is a constant c and an infinite computable sequence of natural numbers $s_0 < s_1 < \cdots$ such that

$$K(A\upharpoonright s_i) \geq s_i + c$$

for all $i \in \mathbb{N}$. Then A is 1-random

Proof Assume A is not 1-random. that is for each $d \in \mathbb{N}$ there is $n \in \mathbb{N}$ s.t

$$K(A\upharpoonright n) \leq n - d$$

Consider the machine M: input μ, it search for $\nu < \mu$, computes $U(\nu)$

Once $U(\nu) = \tau$, it computes $|\tau|$, and $s_{|\tau|}$, and ρ s.t. $\nu\rho = \mu$

and check if $|\tau\rho| = s_{|\tau|}$ if so output $\delta = \tau\rho$

clearly, M is prefix-free

Let c_M be one of its coding constant

Then, For each $i \in \mathbb{N}$,

$$\begin{aligned} K(A\upharpoonright s_i) &\leq K_M(A\upharpoonright s_i) + c_M \\ &= |(A\upharpoonright i)^*| + (s_i - i) + c_M \\ &= K(A\upharpoonright i) + (s_i - i) + c_M \end{aligned}$$

Let $d = c_M - c + 1$, let n be s.t. $K(A\upharpoonright i) \leq n - d$

Then $K(A\upharpoonright n) \leq (n-d) + (s_n - n) + c_M$

$$= s_n - d + c_M = s_n + c - 1 < s_n + c \quad \Rightarrow\Leftarrow \quad \square$$

We now turn to another approach to randomness

von Mises: random sequence should satisfies law of large numbers:

i.e. $\lim_{n\to\infty}\left(\sum_{i=0}^{n} A(i) / n\right) = 1/2$

But 010101... is not random

von Mises: For every selection function $f: \mathbb{N} \to \mathbb{N}$ s.t. $f(0) < f(1) < \cdots$

$$\lim_{n\to\infty} \left(\sum_{i=0}^{n} A(f(i)) / n \right) = 1/2$$

But what is selection function?

Church: $f: \mathbb{N} \to \mathbb{N}$ is a selection function if f is computable increasing function

Def $A \in 2^\omega$ satisfies Church stochastic if for each computable increasing function $f: \mathbb{N} \to \mathbb{N}$, we have $\lim_{n\to\infty} \left(\sum_{i=0}^{n} A(f(i)) / n \right) = 1/2$

Ville: There is a church stochastic $A \in 2^\omega$ s.t. $\sum_{i=0}^{n} A(i)/n \leq \frac{1}{2}$ for all n

This is intuitively not random

Ville proposed that random sequence should also satisfies law of iterated logarithm.

But one cannot guarantee that such definition is sufficient.

Martin Löf proposed a notion of arbitrary test, a sequence is random iff it can past all such tests

Example 1 a random sequence should not be s.t. "every 2^k position is 0"

To test if A is random, we check at each time, e.g. n, that

if every $A(2k) = 0$ for $k < n$

if some $A(2k) = 1$, then A past the test once for all

if $A(2k) = 0$ for all $k < 2n$, Since most sequence should already safe but A is still not (the probability is 2^{-n}), we have strong reason to suspect that A is not random

So for a test to "capture" a non-random sequence, the test should be more and more "accurate", i.e.

at each stage n, most of the sequence should be safe, and only, say 2^{-n} many sequences are still remain

Example 2

If A is random, then the number of 0 and number of 1 should not be 1:2

at stage 1, 1 1 1 3/8 tails

1 1 0

at stage 1, 1 1 1 3/8 fails

1 1 0
1 0 1
1 0 0
0 1 1
0 1 0
0 0 1
0 0 0

generally, at stage n, the probability of the number of 0 is n, the number of 1 is $2n$ is

$$\frac{C_{3n}^n}{2^{3n}} = \frac{(3n)!}{(3n-n)!\,n!} \cdot \frac{1}{2^{3n}} = \frac{(3n)!}{(2n)!\,n!} \cdot \frac{1}{2^{3n}}$$

Note $\lim\limits_{n\to\infty} \frac{C_{3n}^n}{2^{3n}} = 0$

so the test is more and more accurate!

Recall, we say $U \subseteq 2^\omega$ is open, iff there is a set $R \subseteq 2^{<\omega}$ s.t. $U = \bigcup_{\sigma \in R} [\sigma]^<$

Note if U is open, then $U = \bigcup_{\sigma \in R_u} [\sigma]^2$ where R_u is prefix-free

We call R_u a basis of U.

We say an open set U is c.e. if one of its bases R_u is c.e.

We say a sequence of open sets $\{U_n\}_{n\in\omega}$ is uniformly c.e. if

$\{(n,\sigma) \mid \sigma \in R_{U_n}\}$ is c.e.

Def (Martin Löf Random)

1) A martin Löf test is a uniformly c.e. sequence of open sets $\{U_n\}_{n\in\omega}$ such that $\mu(U_n) \leq 2^{-n}$ for all n

2) A sequence A fails the test $\{U_n\}_{n\in\omega}$ if $A \in \bigcap_n U_n$, o.w. A pass the test

3) A sequence A is martin-Löf random if it pass all martin-Löf test

Note we can replace the definition of Martin Löf test by require there is $f: \omega \to \mathbb{Q}^+$ s.t. $\mu(U_n) \leq f(n)$

we can also require Martin Löf test to be $U_0 \supseteq U_1 \supseteq U_2 \cdots$

In the above example the test is: $U_n = \{A \in 2^\omega \mid A(2^k) = 0 \text{ for } k < n\}$

We show that the two approaches are equivalent:

Lemma Let M be a prefix-free machine. Let $k \in \mathbb{N}$, and

$$S = \{ \sigma \mid K_M(\sigma) \leq |\sigma| - k \}. \text{ Then}$$

(1) $\mu(\llbracket S \rrbracket) \leq 2^{-k} \mu(\text{dom} \llbracket M \rrbracket)$ (2) Furthermore,

$\mu(\llbracket S \rrbracket)$ is computable in $\mu(\llbracket \text{dom } M \rrbracket)$ uniformly in k

Proof (1) For each $\sigma \in S$, there is σ^* s.t. $M(\sigma^*) = \sigma$ and $|\sigma^*| \leq |\sigma| - k$

So $\mu(\llbracket S \rrbracket) = \sum_{\sigma \in S} 2^{-|\sigma|} \leq \sum_{\sigma \in S} 2^{-|\sigma^*| - k} = 2^{-k} \sum_{\sigma \in S} 2^{-|\sigma^*|}$

$$\leq 2^{-k} \sum_{\tau \in \text{dom} M} 2^{-|\tau|} = 2^{-k} \mu \llbracket \text{dom} M \rrbracket$$

(2) To compute $\mu \llbracket S \rrbracket$, that is fix ε, fix a rational q, we should decide if $|q - \mu \llbracket S \rrbracket| < \varepsilon$

Now fix 2^{-c} find a finite set $F \subseteq \text{dom} M$ s.t.

$$\mu \llbracket \text{dom } M \rrbracket - \mu \llbracket F \rrbracket < 2^{-c+k}$$

Let N_0 be the machine $M \upharpoonright F$ and N_1 be the machine $M \upharpoonright (\text{dom} M \setminus F)$

Let $T_i = \{ \sigma \mid K_{N_i}(\sigma) \leq |\sigma| - k \}$ $(i = 0, 1)$

Note T_0 is finite so we can compute $\mu(\llbracket T_0 \rrbracket)$

and $\mu \llbracket \text{dom } N_1 \rrbracket < 2^{-c+k}$

so $\mu \llbracket T_1 \rrbracket \leq 2^{-k} \cdot \mu \llbracket \text{dom } N_1 \rrbracket < 2^{-c}$

Since $S = T_0 \cup T_1$ so

$$\mu(\llbracket S \rrbracket) - \mu \llbracket T_0 \rrbracket \leq \mu \llbracket T_1 \rrbracket < 2^{-c} \qquad \square$$

Theorem A is Martin-Löf random iff A is 1-random

Proof $(\Rightarrow)$ Let $U_k = \{ A \mid \exists n \ K(A \upharpoonright n) \leq n - k \}$

U_k is uniformly c.e.

and $\mu(U_k) \le 2^{-k} \cdot \Omega \le 2^{-k}$. so U_k is a Martin-Löf test

Note if A is martin-Löf random, then A pass the test, i.e. $A \notin \bigcap_k U_k$

so there is k s.t. $\forall n\ K(A\upharpoonright n) > n-k$

($\Leftarrow$) Assume A is not M-L random, and test $\{U_k\}_{k\in\mathbb{N}}$ "capture"s A

Let R_k be an prefix-free set s.t. $U_k = \bigcup_{\sigma\in R_k} [\sigma]^{\prec}$

Let $R_k = \{\sigma_i^k : i < N_k\}$ where $N_k \le \omega$

Consider the request $\{(|\sigma_i^k| - k + 1, \sigma_i^k) \mid k \in \mathbb{N}, i < N_k\}$

We can assume $\mu(U_k) \le 2^{-2k}$.

So $\sum_{\substack{k\in\mathbb{N}\\ i<N_k}} (|\sigma_i^k| - k + 1) \le 1$

So there is a prefix-free Machine M fullfill the request

Fix $b \in \mathbb{N}$. Let $k = b + c_M + 1$, since $A \in U_k$

there is $\sigma_i^k \prec A$,

Then $K(\sigma_i^k) \le |\sigma_i^k| - (b + c_M + 1) + c_M < |\sigma_i^k| - b$

so A is not 1-random $\square$