# 09. Construct a prefix-free machine

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There is a natural measure defined on sets of reals (infinite sequences)

for example, for each $\sigma \in 2^{<\omega}$, we define $[\![\sigma]\!] = \{ z \in 2^{\omega} \mid z \succ \sigma \}$

That is the set of infinite extensions of $\sigma$

And we define the measure of $[\![\sigma]\!]$, $\mu([\![\sigma]\!]) = 2^{-|\sigma|}$

An open set $U \subseteq 2^{\omega}$ can be represented by a prefix-free basis $B \subseteq 2^{<\omega}$

namely, $U = \bigcup_{\sigma \in B} [\![\sigma]\!]$

Proof. Let $B_0 = \{ \sigma \in 2^{<\omega} \mid [\![\sigma]\!] \subset U \}$. Let $\{\sigma_0, \sigma_1 \cdots\}$ be the enumeration of $B_0$

Then $U = \bigcup_{i \in \omega} [\![\sigma_i]\!]$

Let $\tau_0 = \sigma_0$, $\tau_{n+1}$ = the first $\sigma_i$ s.t. $\{\tau_0 \cdots \tau_n, \sigma_i\}$ is prefix-free (if no such $\sigma_i$, let $N = n+1$)

If $\tau_n$ is defined for every $n$, let $N = \omega$

Claim $U = \bigcup_{n \in N} \tau_n$ $\square$

So, we can define $\mu(U) = \sum_{i \in N} \mu(\tau_n) = \sum_{i \in N} 2^{-|\tau_n|}$

Note $\mu(\emptyset) = 0$, $\mu(2^{\omega}) = 1$

Clearly, for each prefix-free machine $M$, the halting probability of $M$

is defined as $\sum_{\sigma \in \mathrm{dom}\, M} 2^{-|\sigma|} = \mu(\{ z \mid \exists n\ M(z \upharpoonright n)\downarrow \}) \leq 1$

Therefore $\sum_{\sigma \in 2^{<\omega}} 2^{-K(\sigma)} = \sum_{\tau \in \mathrm{dom}(U)} 2^{-|\tau|} \leq 1$

Define $\Omega = \sum_{\sigma \in 2^{<\omega}} 2^{-K(\sigma)}$ which is the halting probability of the universal prefix-free machine

It is a real number in $(0,1)$, remember it!

We show how to construct a prefix-free machine to meet a given set of requests

Theorem Let $\{ (d_i, \tau_i) \mid i \in \omega \}$ be a computable set of pairs with each $d_i \in \mathbb{N}$, $\tau_i \in 2^{<\omega}$, such that $\sum_{i \in \omega} 2^{-d_i} \leq 1$. (Such sequences are called requests). Then there is a prefix-free machine $M$ meets the requests, i.e. there are strings $\{\sigma_i : i \in \omega\}$ s.t. $\mathrm{dom}\, M = \{\sigma_i\}_{i \in \omega}$, $M(\sigma_i) = \tau_i$ and $|\sigma_i| = d_i$ for all $i \in \omega$

Furthermore, an index of $M$ can be obtained from an index of the request set effectively.

Proof Let $\{(d_i, \tau_i) \mid i \in \omega\}$ be a request

We define an c.e. prefix-free set $\{\sigma_i : i \in \omega\}$ s.t. $|\sigma_i| = d_i$

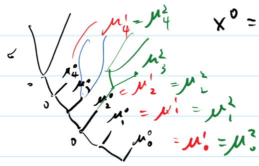
Let $\sigma_0 = 0^{d_0}$

$x^0 = \langle x^0_m : m \in \omega \rangle$

s.t. $x^0_m = 1$ for $m < d_0$

$d_0 = 5$, $\sigma_0 = 00000$ $\quad 2^{-d_0} = 0.0000011111$

$= 0.00001$




$x^0 = 1 - 2^{-d_0} = 0.11111$

$x^0_0 \quad x^0_1 \quad x^0_4$

If $d_1 = 3 \in \{0,1,2,3,4\} = \{m \mid x^0_m = 1\}$, Let $\sigma_1 = \mu^0_3$

and $x^1 = 0.11101$

If $d_2 = 3 \notin \{ m \mid x^1_m = 1 \}$

$X = (x_m : m \in \omega)$

s.t. $x^0_m = 1$ for $m < d_0$

Let $\mu^0_m = 0^{m-1}1$

Note $|\mu^0_m| = m$

$S_0 = \{\delta_0\} \cup \{\mu^0_m : x^0_m = 1\}$ is prefix-free

And $\mu[\![S_0]\!] = 1$

and $x^1 = 0.11101$

If $d_2 = 3 \notin \{m \mid x^1_m = 1\}$

take $j = 2$ the largest such $j < d_2$ and $x^1_j = 1$

Let $\delta_2 = \mu^1_2 0 = \mu^1_j 0^{d_2 - j}$

$x^2 = 0.11011$

For the case of $n+1$ we have $\{\delta_0, \cdots \delta_n\}$, $x^n$, $\{\mu^n_m \mid x^n_m = 1\}$ defined

Subcase 1 $x^n_{d_{n+1}} = 1$

Let $\delta_{n+1} = \mu^n_{d_{n+1}}$. Note $|\delta_{n+1}| = |\mu^n_{d_{n+1}}| = d_{n+1}$

Let $x^{n+1}$ be s.t. $x^{n+1}_m = x^n_m$ for $m \neq d_{n+1}$, $x^{n+1}_{d_{n+1}} = 0$

Let $\mu^{n+1}_m = \mu^n_m$ for $x^{n+1}_m = 1$

Let $S_{n+1} = \{\delta_i : i \leq n+1\} \cup \{\mu^{n+1}_m : x^{n+1}_m = 1\}$,

then $\mu[\![S_{n+1}]\!] = 1$

Subcase 2 $x^n_{d_{n+1}} = 0$

Find the largest $j < d_{n+1}$ s.t. $x^n_j = 1$ —

If such $j$ do not exists. i.e. $x^n_j = 0$ for $j \leq d_{n+1}$

Then $S_n = \{\delta_i \mid i \leq n\} \cup \{\mu^n_m \mid x^n_m = 1 \wedge m > d_{n+1}\}$ finite

$1 = \mu[\![S_n]\!] < \mu[\![\delta_i]\!]_{i \leq n} + \underbrace{\sum_{m > d_{n+1}}}_{\text{infinite } m\text{s}} 2^{-m} \leq \sum_{i \leq n+1} 2^{-d_i} < \sum_{i \in \omega} 2^{-d_i}$

Let $\delta_{n+1} = \mu^n_j 0^{d_{n+1} - j}$. Note $|\delta_{n+1}| = |\mu^n_j| + d_{n+1} - j = d_{n+1}$

Let $x^{n+1}$ be s.t.
$$\begin{cases} x^{n+1}_m = x^n_m & \text{for } m < j \text{ and } m > d_{n+1} \\ x^{n+1}_m = 1 & \text{for } j < m \leq d_{n+1} \\ x^{n+1}_j = 0 & \text{for} \end{cases}$$

Let $\mu^{n+1}_m \quad {}^{n}_{m}$ for $m < j$ and $m > d_{n+1}$

$\mu^{n+1}_m = \mu^n_j 0^{m-j-1} 1$ for $j < m \leq d_{n+1}$ note $|\mu^{n+1}_m| = |\mu^n_j| + (m-j-1) + 1 = m$

Similarly. $S_{n+1} = \{\delta_i \mid i \leq n+1\} \cup \{\mu^{n+1}_m \mid x^{n+1}_m = 1\}$

and $\mu[\![S_{n+1}]\!] = 1$ $\square$