Forcing Truth Lemma

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Let M be c.t.m. for ZF. Let $\mathbb{P} \in M$ be forcing poset. Let $\varphi(\vec{\tau})$ be a sentence of $\mathcal{FL}_{\mathbb{P}} \cap M$ and Let G be $\mathbb{P}$-generic over M. Then

$$M[G] \models \varphi(\vec{\tau}_G) \iff \exists p \in G \ (p \Vdash \varphi(\vec{\tau}))$$

Proof We proof by induction on φ that $\iff$ holds

1. For atomic $\varphi(\vec{\tau})$, we prof by induction on the well-founded relation of $(\tau_1, \theta_1, \in/=) \lhd (\tau_2, \theta_2, \in/=)$

1.1. For $\tau = \theta$.

1.1.1 we show $M[G] \models \tau_G = \theta_G \Rightarrow \exists p \in G \ (p \Vdash \tau = \theta)$

[Exe For each atomic formula φ

$\{p \in \mathbb{P} \mid p \Vdash \varphi \ \vee p \Vdash \neg\varphi\}$ is dense

Hint: $p \Vdash \varphi$ iff $\forall q \leq p \ (q \not\Vdash \neg\varphi)$]

Assume to contradiction that $\forall p \in G \ \ p \not\Vdash \tau = \theta$.

Intuitively, consider the set $D' = \{p \in \mathbb{P} \mid p \Vdash \tau = \theta \ \vee \ p \Vdash \tau \neq \theta\}$

Then D' is dense, therefore $G \cap D' \neq \emptyset$ take $p \in G \cap D'$,

since by assumption $p \not\Vdash \tau = \theta$, then $p \Vdash \tau \neq \theta$, But we can't reach

$M[G] \models \tau_G \neq \theta_G$ by induction

But Consider the set

$D = \{p \in \mathbb{P} \mid p \Vdash \tau = \theta \ \vee$ p witness some weaker condition doesn't force $\tau = \theta\}$

$= \{p \in \mathbb{P} \mid p \Vdash \tau = \theta \ \vee \ \exists \sigma \in \operatorname{dom}\tau \cup \operatorname{dom}\theta \ [(p \Vdash \sigma \in \tau \wedge p \Vdash \sigma \notin \theta) \vee (p \Vdash \sigma \notin \tau \wedge p \Vdash \sigma \in \theta)]\}$

Clearly D is dense as D' is

So let $p \in D \cap G$ and since $p \not\Vdash \tau = \theta$, we have some $\sigma \in \operatorname{dom}\tau \cup \operatorname{dom}\theta$

and we may asume $p \Vdash \sigma \in \tau$ and $p \Vdash \sigma \notin \theta$ (the other case is similar)

Now, by induction hypothesis, $M[G] \models \sigma_G \in \tau_G$

and $M[G] \models \sigma_G \notin \theta_G$ [for if $M[G] \models \sigma_G \in \theta_G$, then there is $q \in G$ s.t. $q \Vdash \sigma \in \theta$,

then there is $r \leq q$, $r \leq p$ s.t. $r \Vdash \sigma \in \theta$. $\rightarrow\leftarrow$ $p \Vdash \sigma \notin \theta \iff \forall r \leq p \ r \not\Vdash \sigma \in \theta$]

Therefore $M[G] \models \tau_G \neq \theta_G$ $\rightarrow\leftarrow$

1.1.2 Assume there is $p \in G$ s.t. $p \Vdash \tau = \theta$, show $M[G] \models \tau_G = \theta_G$

We show $\tau_G \subseteq \theta_G$, the other direction is similar.

take and $x \in \tau_G$ But definition, there is $(\sigma, r) \in \tau$ s.t. $r \in G$ and $x = \sigma_G$

By IH, $\sigma_G \in \tau_G$ implies there is $p_0 \in G$ s.t. $p_0 \Vdash \sigma \in \tau$

Take $q \leq p_0, p$ and $q \in G$, and by the definition of $p \Vdash \tau = \theta$, $q \Vdash \sigma \in \theta$

By IH, $\sigma_G \in \theta_G$ Thus $\tau_G \subseteq \theta_G$

1.2 For $\tau \in \tau$

By IH, $\delta_G \in U_G$. Thus $\tau_G \subseteq U_G$

1.2 For $\pi \in \tau$

1.2.1 We show $M[G] \models \pi_G \in \tau_G \Rightarrow \exists p \in G\ p \Vdash \pi \in \tau$

Again, assume to contradiction that $\forall p \in G\ p \nVdash \pi \in \tau$

Intuitively, $D' = \{p \in P \mid p \Vdash \pi \in \tau \vee p \Vdash \pi \notin \tau\}$ is dense, but still it doesn't work

Let $D = \{p \in P \mid p \Vdash \pi \in \tau \vee$ p nor some weaker condition doesn't force $\pi \in \tau\}$

$= \{p \in P \mid p \Vdash \pi \in \tau \vee \forall q \le p\ \forall (\delta, r) \in \tau\ (q \le r \to q \nVdash \pi = \delta)\}$

Clearly D is dense. so let $p \in G \cap D$, so we have

$$\forall (\delta, r) \in \tau\ \forall q \le p, r\ \ q \nVdash \pi = \delta$$

Take any $x \in \tau_G$, there is some $(\delta, r) \in \tau$ s.t. $\delta_G = x$ and $r \in G$

Let $q_0 \le p, r$ and $q_0 \in G$, then $\forall q \le q_0\ q \nVdash \pi = \delta$ i.e. $q_0 \Vdash \pi \neq \delta$

By IH, $M[G] \models x = \delta_G \neq \pi_G$ so $M[G] \models \pi_G \notin \tau_G$ $\Rightarrow\Leftarrow$

1.2.2 To show $\exists p \in G\ p \Vdash \pi \in \tau \Rightarrow M[G] \models \pi_G \in \tau_G$

By definition $E = \{q \le p \mid \exists (\delta, r) \in \tau\ [q \le r \wedge q \Vdash \pi = \delta]\}$ is dense below p

and since $p \in G$, $E \cap G \neq \emptyset$, take q and $(\delta, r) \in \tau$ s.t. $q \le r$ and $q \Vdash \pi = \delta$

then $r \in G$ so $\delta_G \in \tau_G$, and by IH, $\pi_G = \delta_G$, so $\pi_G \in \tau_G$

1.3 For $\neg\varphi$

1.3.1 To show $M[G] \models \neg\varphi \Rightarrow \exists p \in G\ p \Vdash \neg\varphi$

Assume $M[G] \not\models \varphi$. by IH $\forall p \in G\ p \nVdash \varphi$

Assume to contradiction that $\forall p \in G\ p \nVdash \neg\varphi$

Let $D = \{p \in P \mid p \Vdash \varphi \vee p \Vdash \neg\varphi\}$, then D is dense

so there is $p \in D \cap G$ $\Rightarrow\Leftarrow$

1.3.2 To show $\exists p \in G\ p \Vdash \neg\varphi \Rightarrow M[G] \models \neg\varphi$

Assume to contradiction that $M[G] \models \varphi$. By IH, $\exists p' \in G\ p' \Vdash \varphi$

take $r \le p, p'$ then contradict with $p \Vdash \neg\varphi$

1.4 $\varphi \wedge \psi$ is trivial

1.5 For $\forall x \varphi(x)$

1.5.1 To show $M[G] \models \forall x \varphi(x) \Rightarrow \exists p \in G\ p \Vdash \forall x \varphi(x)$

Assume to contradiction that $\forall p \in G\ p \nVdash \forall x \varphi(x)$

Consider $D = \{ p \in \mathbb{P} \mid p \Vdash \forall x\, \varphi(x) \vee \exists \tau \in M^{\mathbb{P}}\ p \Vdash \neg\varphi(\tau) \}$

Then D is dense. Let $p \in D \cap G$, since $p \not\Vdash \forall x\, \varphi(x)$,

there is $\tau \in M^{\mathbb{P}}$ s.t. $p \Vdash \neg\varphi(\tau)$. By IH. $M[G] \models \neg\varphi(\tau_G)$ $\mapsto\!\leftarrow\!|$

1.5.2 To show $\exists p \in G\ p \Vdash \forall x\, \varphi(x) \Rightarrow M[G] \models \forall x\, \varphi(x)$

$\forall x \in M[G]$, $x = \tau_G$ for some $\tau \in M^{\mathbb{P}}$, but then $p \Vdash \varphi(\tau)$

so $M[G] \models \varphi(\tau_G)$ by IH. $\square$