Set Theory II

集合论 II

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Previously on Set Theory

- the Truth Lemma
- $M[G] \vDash \mathrm{ZFC}$
- c.c.c. implies cardinality preserving

Computing cardinal exponentiation

Recall:

$$\mathrm{Fn}(\omega_2^M \times \omega, 2)$$

$$= \left\{ p \,\middle|\, \text{ is a finite partial function from } \omega_2^M \times \omega \text{ to } 2 \right\}$$

Computing cardinal exponentiation

Fact

Let $\mathbb{P} = \mathrm{Fn}(\omega_2^M \times \omega, 2)$. Then

- $\mathbb{P}$ has c.c.c.
- $(\mathrm{card}\,\mathbb{P} = \aleph_2)^M$

From now on, let $\mathbb{P} = \mathrm{Fn}(\omega_2^M \times \omega, 2)$

Computing cardinal exponentiation

Let G be $\mathbb{P}$-generic over M. Then in $M[G]$:

- Let $g = \bigcup G$ is a function from $\omega_2 \times \omega$ to 2
- Define $g_\alpha : \omega \to 2$ for $\alpha < \omega_2$ such that

$$g_\alpha(n) = g(\alpha, n)$$

- $g_\alpha \neq g_\beta$ if $\alpha \neq \beta$

EXE: For any $r \in (2^\omega)^M$, $r \neq g_\alpha$ for any $\alpha < \omega_2^M$

Computing cardinal exponentiation

Therefore in $M[G]$, $2^{\aleph_0} \geq \aleph_2$

Note: $\aleph_2$ can be replaced by any aleph

Computing cardinal exponentiation

But we can also compute the exact aleph of $2^{\aleph_0}$

Let M be a c.t.m for $\mathrm{ZFC} + V = L$, we show

$$M[G] \models 2^{\aleph_0} = \aleph_2$$

Computing cardinal exponentiation

Definition (Nice name)

For $\tau \in V^{\mathbb{P}}$, a nice name for a subset of τ is a name of the form

$$\bigcup \left\{ \{\sigma\} \times A_\sigma \;\middle|\; \sigma \in \operatorname{dom} \tau \right\}$$

where each A_σ is an antichain in $\mathbb{P}$

Computing cardinal exponentiation

Lemma

Fix $\tau \in V^{\mathbb{P}}$. Let $\lambda = \operatorname{dom} \tau$ be infinite. Then there are no more than $\aleph_2^{\lambda}$ nice name for subset of τ

Computing cardinal exponentiation

Lemma

Fix $\tau, \mu \in M^{\mathbb{P}}$. There is a nice name $\vartheta \in M^{\mathbb{P}}$ for a subset of τ such that

$$\mathbb{1} \Vdash (\mu \subset \tau \to \mu = \vartheta)$$

Proof.

Let $\vartheta = \bigcup \left\{ \{\sigma\} \times A_\sigma \;\middle|\; \sigma \in \mathrm{dom}(\tau) \right\}$ where each A_σ is maximal such that A_σ is an antichain and

$$\forall p \in A_\sigma \; p \Vdash \sigma \in \mu$$

Computing cardinal exponentiation

Lemma

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Computing cardinal exponentiation

Lemma

$M[G] \vDash 2^{\aleph_0} \leq \aleph_2$

Theorem

$\mathrm{Con}(\mathrm{ZFC}) \to \mathrm{Con}(\mathrm{ZFC} + 2^{\aleph_0} = \aleph_2)$

Conclusion

Why these relative consistency results are provable in a finitistic metatheory?

Thank you