

Set Theory II

集合论 II

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Spring 2016

Previously on Set Theory

Truth Lemma is critical in the proof of $(\text{Separation})^{M[G]}$

Forcing

EXE: Let M be a c.t.m for ZF, show

- $M[G] \models$ Power set axiom
- $M[G] \models$ Replacement
- If $M \models$ AC, then $M[G] \models$ AC

Forcing

We continue the proof of the the truth lemma by induction on $\varphi \in \mathcal{FL}_{\mathbb{P}}^M$ that

$$M[G] \models \varphi(\bar{\sigma}_G) \Leftrightarrow \exists p \in G (p \Vdash \varphi(\bar{\sigma}))$$

Preserving cardinality

Definition

For a forcing poset $\mathbb{P} \in M$, we say

- \mathbb{P} **preserves cardinals** iff for all generic G and all $\beta < o(M)$

$$(\beta \text{ is a cardinal})^M \leftrightarrow (\beta \text{ is a cardinal})^{M[G]}$$

- \mathbb{P} **preserves cofinalities** iff for all generic G and all limit $\gamma < o(M)$

$$\text{cf}^M(\gamma) = \text{cf}^{M[G]}(\gamma)$$

Preserving cardinality

Lemma

For a forcing poset $\mathbb{P} \in M$,

- If for any limit β such that $\omega < \beta < o(M)$ we have

$$(\beta \text{ is regular})^M \leftrightarrow (\beta \text{ is regular})^{M[G]},$$

then \mathbb{P} preserves cofinality

- If \mathbb{P} preserves cofinality, then \mathbb{P} preserves cardinality

Preserving cardinality

Theorem

If $\mathbb{P} \in M$ has $(\text{c.c.c.})^M$, then P preserves cofinalities and cardinality

If $(\mathbb{P} \text{ has c.c.c.})^M$, then P preserves regularities for limit $\omega < \beta < o(M)$

Then $\tilde{F} \in M$

Proof Assume $(\mathbb{P} \text{ is c.c.c.})^M$, show

$$(\beta \text{ is regular})^M \rightarrow (\beta \text{ is regular})^{M[G]}$$

Assume $(\beta \text{ is not regular})^{M[G]}$

$\exists f \in M[G] \quad f: \alpha \rightarrow \beta$ is cofinal

Let $f = \dot{f}_a$ where $\dot{f} \in M^{\mathbb{P}}$

By truth lemma $\exists p \in G$

p.t. $\dot{f}: \check{\alpha} \rightarrow \check{\beta}$ is cofinal

We want to find $F \in M$ witness β is not regular

In M , we know

$$\forall r \in \alpha \quad p \Vdash \dot{f}(r) < \check{\beta}$$

$$\text{Let } \tilde{F}(r) = \left\{ \check{\beta} \mid \exists q \Vdash \dot{f}(r) = \check{\beta} \right\}$$

Next on Set Theory

- Computing cardinal exponentiation in $M[G]$
- Final remarks