Set Theory II

集合论 II

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

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Previously on Set Theory

Truth Lemma is critical in the proof of $(\text{Separation})^{M[G]}$

Forcing

EXE: Let M be a c.t.m for ZF, show

- $M[G] \vDash$ Power set axiom
- $M[G] \vDash$ Replacement
- If $M \vDash$ AC, then $M[G] \vDash$ AC

Forcing

We continue the proof of the the truth lemma by induction on $\varphi \in \mathcal{F}\mathcal{L}_{\mathbb{P}}^{M}$ that

$$M[G] \vDash \varphi(\bar{\sigma}_G) \Leftrightarrow \exists p \in G\ (p \Vdash \varphi(\bar{\sigma}))$$

Preserving cardinality

Definition

For a forcing poset $\mathbb{P} \in M$, we say

- $\mathbb{P}$ preserves cardinals iff for all generic G and all $\beta < o(M)$

$$(\beta \text{ is a cardinal})^M \leftrightarrow (\beta \text{ is a cardinal})^{M[G]}$$

- $\mathbb{P}$ preserves cofinalities iff for all generic G and all limit $\gamma < o(M)$

$$\mathrm{cf}^M(\gamma) = \mathrm{cf}^{M[G]}(\gamma)$$

Preserving cardinality

Lemma

For a forcing poset $\mathbb{P} \in M$,

- If for any limit β such that $\omega < \beta < o(M)$ we have

  $$(\beta \text{ is regular})^M \leftrightarrow (\beta \text{ is regular})^{M[G]},$$

  then $\mathbb{P}$ preserves cofinality
- If $\mathbb{P}$ preserves cofinality, then $\mathbb{P}$ preserves cardinality

Preserving cardinality

Theorem

If $\mathbb{P} \in M$ has $(\text{c.c.c.})^M$, then P preserves cofinalities and cardinality

If $(\mathbb{P}$ has c.c.c.$)^M$, then P preserves regularities for limit $\omega < \beta < o(M)$

Proof. Assume $(\mathbb{P}$ is c.c.c.$)^M$, show

$(\beta \text{ is regular})^M \to (\beta \text{ is regular})^{M[G]}$

Assume $(\beta$ is not regular$)^{M[G]}$

$\exists f \in M[G]$ $f : \alpha \to \beta$ is cofinal

Let $f = \dot{f}_G$ where $\dot{f} \in M^{\mathbb{P}}$

By truth lemma $\exists p \in G$

$p \Vdash \dot{f} : \check{\alpha} \to \check{\beta}$ is cofinal

We want to find $F \in M$ witness β is not regular

In M, we know

$\forall \gamma < \alpha$ $p \Vdash \dot{f}(\check{\gamma}) < \check{\beta}$ ✓

Let $F(\gamma) = \{ \xi < \beta \mid \exists_{q \leq p} q \Vdash \dot{f}(\check{\gamma}) = \check{\xi} \}$

Then $F \in M$

Claim $|F(\gamma)| \leq \aleph_0$.

If $\xi_1, \xi_2 \in F(\gamma)$ and $\xi_1 \neq \xi_2$

then there $q_1, q_2 \leq p$

$q_1 \Vdash \dot{f}(\check{\gamma}) = \xi_1$, $q_2 \Vdash \dot{f}(\check{\gamma}) = \xi_2$

and $q_1 \perp q_2$

$\{ q_i \mid \xi_i \in F(\gamma) \}$ is antichain

which is countable

And $f(\gamma) \in F(\gamma) \subseteq \beta$

$\sup\left(\bigcup_{\gamma < \alpha} F(\gamma)\right) = \beta$

but $\left|\bigcup_{\gamma<\alpha} F(\gamma)\right| = \alpha \cdot \aleph_0 = \alpha$

Next on Set Theory

- Computing cardinal exponentiation in $M[G]$
- Final remarks