Set Theory II

集合论 II

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Previously on Set Theory

- $M[G] = \left\{\tau_G \,\middle|\, \tau \in M^{\mathbb{P}}\right\}$, where $\tau_G = \left\{\sigma_G \,\middle|\, \exists(\sigma, p) \in \tau \ (p \in G)\right\}$
- $\emptyset_G = \emptyset$, $\check{x}_G = x, \dot{G}_G = G$
- Pairing: $\left(\mathrm{up}(\tau, \sigma)\right)_G = \{\tau_G, \sigma_G\}$
- Union: Let $\pi = \left\{(\sigma, p) \in M^{\mathbb{P}} \times \mathbb{P} \,\middle|\, \exists(\theta, q) \in \tau \exists r \in \mathbb{P}\ [p \leq q \wedge p \leq r \wedge (\sigma, r) \in \theta]\right\}$, then $\pi_G = \bigcup \tau_G$

Forcing

To show that Separation holds $M[G]$:

Given $\mathbb{P}$-name τ, σ and a formula φ, we want to cook up a $\mathbb{P}$-name π in the ground model M such that

$$\pi_G = \left\{ x \in \tau_G \;\middle|\; \varphi^{M[G]}(x, \tau_G, \sigma_G) \right\}$$

Forcing

To cook up such a $\mathbb{P}$-name in M, we must be able to talk about the truth of $M[G]$ within M, which we cannot without the information from G

But still, we can define a forcing language and forcing relation within the ground model M to talk about what $M[G]$ must be or might be even without knowing G

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Forcing

Definition

For a forcing poset $\mathbb{P}$, we define the $\mathbb{P}$ forcing language $\mathcal{F}\mathcal{L}_{\mathbb{P}}$ to be the class of logical formulas formed using the binary relation $\in$ and all the $\mathbb{P}$-names as constant symbols

Forcing

We want to define the relation $p \Vdash \varphi$ for $p \in \mathbb{P}$ and $\varphi \in \mathcal{F}\mathcal{L}_{\mathbb{P}}$ such that the following holds

Lemma (The Truth Lemma)

Let M be a c.t.m for ZF, let $\mathbb{P} \in M$ be a forcing poset, let ψ be a sentence of $\mathcal{F}\mathcal{L}_{\mathbb{P}} \cap M$, and let G be $\mathbb{P}$-generic over M. Then

$$M[G] \vDash \psi \Leftrightarrow \exists p \in G \, (p \Vdash \psi)$$

If the definition of forcing works, i.e. assuming the truth lemma we can show

Lemma

The Separation Schema hold in $M[G]$

Proof.

Let

$$\pi = \left\{(\vartheta, p) \;\middle|\; \vartheta \in \operatorname{dom}\tau \wedge p \in \mathbb{P} \wedge p \Vdash \left(\vartheta \in \tau \wedge \varphi(\vartheta, \tau, \sigma)\right)\right\}$$

□

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Forcing

Definition (Forcing (atomic formulas))

For $\mathbb{P}$-names τ, ϑ, π,

- $p \Vdash \tau = \vartheta$ iff

  $\forall \sigma \in \operatorname{dom} \tau \cup \operatorname{dom} \vartheta \forall q \leq p \left[q \Vdash \sigma \in \tau \leftrightarrow q \Vdash \sigma \in \vartheta \right]$

- $p \Vdash \pi \in \tau$ iff $\left\{ q \leq p \,\middle|\, \exists (\sigma, r) \in \tau [q \leq r \wedge q \Vdash \pi = \sigma] \right\}$ is dense below p

Note: Again, this is a definition by recursion

Forcing

Definition

Let $\mathbb{P}$ be a forcing poset and $p \in \mathbb{P}$. We say $D \subset \mathbb{P}$ is dense below p if for each $q \leq p$ there is a $r \leq q$ such that $r \in D$

Fact

If G is $\mathbb{P}$ generic over M, $p \in G$ and $D \in M$ is dense below p, then $G \cap D \neq \emptyset$

Forcing

Lemma

For atomic formula $\varphi \in \mathcal{F}\mathcal{L}_{\mathbb{P}}$,

- If $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$
- $p \Vdash \varphi$ iff $\left\{r \leq p \,\middle|\, r \Vdash \varphi\right\}$ is dense below p

Note: $\left\{r \leq p \,\middle|\, r \Vdash \varphi\right\}$ is not dense below p iff

$\exists q \leq p \forall r \leq q \; (r \nVdash \varphi)$

Forcing

Lemma

For atomic formula $\varphi \in \mathcal{F}\mathcal{L}_{\mathbb{P}}$,

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Forcing

Definition

For atomic formula $\varphi \in \mathcal{F}\mathcal{L}_{\mathbb{P}}$ and $p \in \mathbb{P}$, $p \Vdash \neg\varphi$ iff

$\forall q \leq p \ (q \nVdash \varphi)$

Lemma

For atomic formula $\varphi \in \mathcal{F}\mathcal{L}_{\mathbb{P}}$ and $p \in \mathbb{P}$, $p \Vdash \varphi$ iff

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Forcing

We continue the definition for complex formulas

Definition

- $p \Vdash \neg\varphi$ iff $\forall q \leq p \; (q \nVdash \varphi)$
- $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$
- $p \Vdash \forall x \varphi(x)$ iff $p \Vdash \varphi(\tau)$ for every $\mathbb{P}$-name τ

Forcing

Lemma

- $p \Vdash \varphi \to \psi$ iff $\neg \exists q \leq p \; (q \Vdash \varphi \wedge q \Vdash \neg\psi)$
- $p \Vdash \varphi \vee \psi$ iff $\left\{q \,\middle|\, (q \Vdash \varphi) \vee (q \Vdash \psi)\right\}$ is dense below p
- $p \Vdash \exists x \varphi(x)$ iff $\left\{q \,\middle|\, \exists \tau \in V^{\mathbb{P}} \, [q \Vdash \varphi(\tau)]\right\}$ is dense below p

Forcing

Lemma

For all $\varphi \in \mathcal{FL}_{\mathbb{P}}$,

- $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$
- $p \Vdash \varphi$ iff $\left\{ q \leq p \,\middle|\, q \Vdash \varphi \right\}$ is dense below p
- $p \Vdash \varphi$ iff $\forall q \leq p \; (q \nVdash \neg\varphi)$

Fact (Forcing relation is absulote)

Let M be a c.t.m for ZF and $\mathbb{P} \in M$. Then $\mathcal{FL}_{\mathbb{P}}^{M} = \mathcal{FL}_{\mathbb{P}} \cap M$, and for any $p \in M$, $\varphi \in \mathcal{FL}_{\mathbb{P}}^{M}$,

$$(p \Vdash \varphi)^{M} \leftrightarrow p \Vdash \varphi$$

Next on Set Theory

- Preserving cardinality
- Consistency of $\neg$GCH