Set Theory II

集合论 II

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

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Previously on Set Theory

- Partial orders
- Antichain, Countable chain condition
- Dense set
- Filter, generic filter

Generic Extensions

Through out this lecture, we fix a c.t.m. M of ZFC and a forcing poset $\mathbb{P} \in M$

Definition

τ is a $\mathbb{P}$-name iff τ is a binary relation and

$$\forall (\sigma, p) \in \tau \ (\sigma \text{ is a } \mathbb{P}\text{-name} \wedge p \in \mathbb{P})$$

Note: This is a definition by recursion

Let $V^{\mathbb{P}}$ be the class of all $\mathbb{P}$-name

Generic Extensions

Fact

Being a $\mathbb{P}$-name is absolute for transitive models of ZF

We define $M^{\mathbb{P}} = V^{\mathbb{P}} \cap M = (V^{\mathbb{P}})^M$

Generic Extensions

What the $\mathbb{P}$-names name:

Definition

Let τ be a $\mathbb{P}$-name and $G \subset \mathbb{P}$. By recursion, we define

$$\tau_G = \left\{\sigma_G \,\middle|\, \exists p \in G[(\sigma, p) \in \tau]\right\}$$

For M a transitive model of ZF and $\mathbb{P} \in M$, define $M[G] = \left\{\tau_G \,\middle|\, \tau \in M^{\mathbb{P}}\right\}$.

Generic Extensions

Example

- $\emptyset$ is a $\mathbb{P}$-name, and $\emptyset_G = \emptyset$ no matter what $\mathbb{P}$ or what G is
- $\tau = \{(\sigma_1, p_1), (\sigma_2, p_2), (\sigma_3, p_3)\}$

Definition (Canonical name)

For each set x, its canonical name is defined as

$$\check{x} =_{\mathrm{df}} \left\{ (\check{y}, \mathbb{1}) \,\middle|\, y \in x \right\}$$

Note: this is also a definition by recursion

Generic Extensions

Lemma

Let G be a filter on $\mathbb{P}$. Then

- $\forall x \in M\ [\check{x} \in M^{\mathbb{P}} \wedge \check{x}_G = x]$
- $M \subset M[G]$

Generic Extensions

Normally, we do not expect such generic $G \in M$

Lemma

If $\mathbb{P} \in M$ is atomless (namely, there is no $r \in \mathbb{P}$ such that for any $p, q \le r$, $p \not\perp q$) and the filter G is $\mathbb{P}$-generic over M, then $G \notin M$

Generic Extensions

But people in M can still "talk about" G:

Definition

Define $\dot{G} = \left\{ (\check{p}, p) \;\middle|\; p \in \mathbb{P} \right\}$

Fact

$\dot{G}_G = G \in M[G]$

Generic Extensions

Usually, $M[G]$ is strictly bigger than M, but not that much

Lemma

- $\mathrm{rank}(\tau_G) \leq \mathrm{rank}(\tau)$ for all τ
- $\mathrm{OR} \cap M[G] = \mathrm{OR} \cap M$
- $\mathrm{card}\, M[G] = \mathrm{card}\, M$ (M is transitive model of ZF, but can be uncountable)

Generic Extensions

Lemma

If N is a transitive model of ZF with $M \subset N$ and $G \in N$, then $M[G] \subset N$

Generic Extensions

We want to show that $M[G]$ is actually a model of ZFC if M is

Definition

- $\mathrm{up}(\sigma, \tau) =_{\mathrm{df}}= \{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}$
- $\mathrm{op}(\sigma, \tau) =_{\mathrm{df}}= \mathrm{up}\Big(\mathrm{up}(\sigma, \sigma), \mathrm{up}(\sigma, \tau)\Big)$

Generic Extensions

Lemma

Let G be a filter on $\mathbb{P}$, $\sigma, \tau \in M^{\mathbb{P}}$, then

- $\mathrm{up}(\sigma, \tau), \mathrm{op}(\sigma, \tau) \in M^{\mathbb{P}}$ and
- $\mathrm{up}_G(\sigma, \tau) = \{\sigma_G, \tau_G\}$,
- $\mathrm{op}_G(\sigma, \tau) = (\sigma_G, \tau_G)$

Generic Extensions

Lemma

$M[G]$ is transitive and is a model for axioms of extensionality, foundation, pairing, and union

$M[G]$ is transtive, so is a model of

extensionality

Foundation

Pairing

Union

Let $\pi = \{ (\delta, p) \in M^{\mathbb{P}} \mid \exists (\theta, q) \in \tau \ \exists r \in \mathbb{P} \ (p \leq q \wedge p \leq r \wedge$
$(\delta, r) \in \theta) \}$

Then $\bigcup \tau_a = \pi_a$

Generic Extensions

Until new we only require G being a filter. We need G to be generic so that the extension $M[G]$ is closed under complements, so that Separation Schema holds in $M[G]$

Fact

If G is $\mathbb{P}$-generic over M, then $(\mathbb{P} \setminus G) \in M[G]$

Forcing

Definition

For a forcing poset $\mathbb{P}$, we define the $\mathbb{P}$ forcing language $\mathcal{F}\mathcal{L}_{\mathbb{P}}$ to be the class of logical formulas formed using the binary relation $\in$ and all the $\mathbb{P}$-names as constant symbols

Forcing

Now we define the notion $p \Vdash \varphi$ for $p \in \mathbb{P}$ and $\varphi \in \mathcal{F}\mathcal{L}_{\mathbb{P}}$

Definition (Forcing (atomic formulas))

For $\mathbb{P}$-names τ, ϑ, π,

- $p \Vdash \tau = \vartheta$ iff

  $\forall \sigma \in \operatorname{dom} \tau \cup \operatorname{dom} \vartheta \forall q \leq p \left[q \Vdash \sigma \in \tau \leftrightarrow q \Vdash \sigma \in \vartheta \right]$
- $p \Vdash \pi \in \tau$ iff $\left\{ q \leq p \,\middle|\, \exists (\sigma, r) \in \tau [q \leq r \wedge q \Vdash \pi = \sigma] \right\}$ is dense below p

Note: Again, this is a definition by recursion

Lemma

For atomic formula $\varphi \in \mathcal{F}\mathcal{L}_{\mathbb{P}}$,

- If $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$
- $p \Vdash \varphi$ iff $\left\{ r \leq p \;\middle|\; r \Vdash \varphi \right\}$ is dense below p

Forcing

Definition

For atomic formula $\varphi \in \mathcal{FL}_{\mathbb{P}}$ and $p \in \mathbb{P}$, $p \Vdash \neg\varphi$ iff

$\neg\exists q \leq p \ (q \Vdash \varphi)$

Lemma

For atomic formula $\varphi \in \mathcal{FL}_{\mathbb{P}}$ and $p \in \mathbb{P}$, $p \Vdash \varphi$ iff

$\neg\exists q \leq p \ (q \Vdash \neg\varphi)$

Next on Set Theory

- Continue definition of forcing
- Truth lemma