

Set Theory II

集合论 II

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Spring 2016

Previously on Set Theory

- By method of inner models, we cannot prove the consistency of $\neg\text{CH}$
- “Outer model” method
 - Countable transitive model
 - Forcing

The semantics of classical logic is two-valued, namely

True / False

When we want to talk about possibilities / imaginations,
we want more

Partial orders

Recall:

A **partial order** is an ordered-pair (\mathbb{P}, \leq) , where \leq is a transitive, reflexive binary relation on \mathbb{P} such that

$$\forall p, q \in \mathbb{P} [p \leq q \rightarrow q \leq p \rightarrow p = q]$$

Partial orders

For convenience, we assume every partially ordered set has a largest element.

Convention

- A **forcing poset** (partially ordered set) is a triple $(\mathbb{P}, \leq, \mathbf{1})$ such that (\mathbb{P}, \leq) is a partial order and $\mathbf{1} \in \mathbb{P}$ is the largest element
- Elements of \mathbb{P} are called **forcing conditions**
- $p \leq q$ is read “ p **extends** q ” or “ p is **stronger** than q ”

Partial orders

Definition

Let \mathbb{P} be a poset. We say

- $p, q \in \mathbb{P}$ are **compatible** (written $p \not\perp q$) if they have a common extension, i.e. there is an $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$
- p, q are **incompatible** (written $p \perp q$) if they are not compatible

Partial orders

Definition

Let \mathbb{P} be a poset.

- A **antichain** of \mathbb{P} is a subset $A \subset \mathbb{P}$ whose elements are pairwise incompatible
- We say \mathbb{P} has the **countable chain condition (ccc)** iff every antichain of \mathbb{P} is countable

Partial orders

Example

Consider a subtree $\mathbb{P} \subset 2^{<\omega}$. Conditions in \mathbb{P} are finite $\{0, 1\}$ sequences, and $p \leq q$ iff p is an end-extension of q .

- $p \not\leq q$ iff $p \leq q$ or vice versa
- \mathbb{P} has the ccc, since \mathbb{P} is at most countable

How about $\mathbb{P} \subset 2^{<\kappa}$ for some uncountable κ ?

Partial orders

Example

For any I, J , let $\text{Fn}(I, J)$ be the set of all finite partial functions from I to J , i.e. $p \in \text{Fn}(I, J)$ iff p is finite, and p is a function with $\text{dom } p \subset I$, $\text{ran } p \subset J$

For $p, q \in \text{Fn}(I, J)$, define $p \leq q$ iff $p \supset q$.

- $p \not\leq q$ iff they agree on $\text{dom } p \cap \text{dom } q$

Partial orders

Lemma

F_n has the ccc if and only if $I = \emptyset$ or J is countable

$F_n(I, J)$

Partial orders

Definition

A family of sets \mathcal{A} forms a **delta system** with **root** R iff for any $X, Y \in \mathcal{A}$ with $X \neq Y$, we have $X \cap Y = R$

Lemma

Let κ be an **uncountable regular** cardinal, and let \mathcal{A} be a family of **finite** sets with $\text{card } \mathcal{A} = \kappa$. Then there is a $\mathcal{B} \in \underbrace{[\mathcal{A}]^\kappa}$ such that \mathcal{B} forms a delta system. $\mathcal{B} \subseteq \mathcal{A}$ and $|\mathcal{B}| = \kappa$

each $X \in \mathcal{A}$ is finite, $\text{card } A = \kappa$, κ is uncountable regular,

Find a delta system $\mathcal{B} \subset \mathcal{A}$ with $\text{card } \mathcal{B} = \kappa$

Proof

Find $\mathcal{B} \subseteq \mathcal{A}$ s.t. every $X \in \mathcal{B}$, $\text{card } X = n$ for some fixed n .

We prove by induction on n .

If $\text{card } \mathcal{B}_0 = \kappa$ and $\forall X \in \mathcal{B}_0, \text{card } X = n$ then \mathcal{B}_0 contains a delta system of cardinality κ

Case 1 $n=1$, we have a delta system with \emptyset root

Case 2 each $X \in \mathcal{B}_0, |X|=n+1$

Consider $D_p = \{X \in \mathcal{B}_0 \mid p \subset X\}$ for $p \in \mathcal{U}(\mathcal{B}_0)$

Sub case 1

there is $p \in \mathcal{U}(\mathcal{B}_0)$ s.t. $\text{card } D_p = \kappa$

Then $D_p' = \{X \in D_p \mid X \subset p\}$

is a set of cardinality κ , and every element in it has cardinality n

By IH, there is a delta system of cardinality κ s.t. $D_p'' \subseteq D_p'$

$D_p''' = \{X \cup \{p\} \mid X \in D_p''\}$ is a delta system $\subseteq \mathcal{A}$

Sub case 2 every D_p has cardinality $< \kappa$

We construct a sequence of pairwise disjoint sets of length κ

Given $\{X_i \mid i < \alpha\}$ built (w.l.o.g.)

Let $Y = \bigcup_{i < \alpha} X_i$, then $\text{card } Y < \kappa$

and $\text{card } \bigcup_{p \in Y} D_p < \kappa$ (since κ is regular)

Let X_α be the least in $\mathcal{B}_0 \setminus \bigcup_{p \in Y} D_p$

Then $X_\alpha \cap X_i = \emptyset$ for $i < \alpha$

So $\{X_\alpha \mid \alpha < \kappa\}$ is a delta system with \emptyset root \square

Assume $I \neq \emptyset$ and J is countable.

Show that $\text{Fn}(I, J)$ has the ccc

Proof Assume to contradiction that
there is an antichain $A \subseteq \text{Fn}(I, J)$

wt. $|A| = \aleph_0$,

Let $B \subseteq A$ s.t. $|B| = \aleph_0$,

and $\{ \text{dom } p \mid p \in B \}$ is a delta system

with root $s \in I$

i.e. for $p, q \in B$, $\text{dom } p \cap \text{dom } q = s$

Since B is an antichain

P1

Partial orders

Definition

Let \mathbb{P} be a forcing poset. We say $D \subset \mathbb{P}$ is **dense** iff

$$\forall p \in \mathbb{P} \exists q \in D \ q \leq p$$

Partial orders

Example

Assume I is infinite and J is non-empty. Then sets of form $\{q \in \text{Fn}(I, J) \mid i \in \text{dom}(q)\}$ and $\{q \in \text{Fn}(I, J) \mid j \in \text{ran}(q)\}$ are dense whenever $i \in I$ and $j \in J$

Partial orders

Definition

Let \mathbb{P} be a forcing poset. Then $F \subset \mathbb{P}$ is a **filter** on \mathbb{P} iff

- $1 \in F$
- $\forall p, q \in F \exists r \in F (r \leq p \wedge r \leq q)$
- $\forall p, q \in \mathbb{P} (p \geq q \rightarrow q \in F \rightarrow p \in F)$

Partial orders

Lemma

Let \mathbb{P} be a forcing poset, \mathcal{D} be a **countable** family of dense subsets of \mathbb{P} , and fix any $p \in \mathbb{P}$. Then there exists a filter G on \mathbb{P} such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$

In this case, we say G is a generic filter on \mathbb{P} for \mathcal{D}

Intuitively, A generic filter G is what we want to build so that fulfills the **requirements** carried by those dense sets

Partial orders

Lemma

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Find a generic filter G on \mathbb{P} for \mathcal{D}
such that $p \in G$

Generic Extensions

From now on, we assume M is a c.t.m. for ZFC

Convention

When we say \mathbb{P} is a forcing poset in M , we mean $(\mathbb{P}, \leq, \mathbb{1}) \in M$, and M think $(\mathbb{P}, \leq, \mathbb{1})$ is a forcing poset.

Note: Being a forcing poset is absolute for M

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Generic Extensions

Definition

Let \mathbb{P} be a forcing poset in M . We say G is a \mathbb{P} -generic over M iff G is a filter on \mathbb{P} and $G \cap D \neq \emptyset$ for all dense set $D \subset \mathbb{P}$ such that $D \in M$

Generic Extensions

Lemma (Generic Filter Existence)

Let \mathbb{P} be a forcing poset in M . Then for every $p \in \mathbb{P}$, there exists a filter G on \mathbb{P} such that $p \in G$ and G is \mathbb{P} -generic over M

Generic Extensions

Normally, we do not expect such generic $G \in M$

Lemma

If $\mathbb{P} \in M$ is **atomless** (namely, there is no $r \in \mathbb{P}$ such that for any $p, q \leq r$, $p \not\perp q$) and the filter G is \mathbb{P} -generic over M , then $G \notin M$

Next on Set Theory

- \mathbb{P} -names
- Generic Extension