Set Theory II

集合论 II

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

Spring 2016

Previously on Set Theory

- By method of inner models, we cannot prove the consistency of $\neg$CH
- "Outer model" method
  - Countable transitive model
  - Forcing

The semantics of classical logic is two-valued, namely

True / False

When we want to talk about possibilities / imaginations,

we want more

Partial orders

Recall:

A partial order is an ordered-pair $(\mathbb{P}, \leq)$, where $\leq$ is a transitive, reflexive binary relation on $\mathbb{P}$ such that

$$\forall p, q \in \mathbb{P}[p \leq q \rightarrow q \leq p \rightarrow p = q]$$

Partial orders

For convenience, we assume every partially ordered set has a largest element.

Convention

- A forcing poset (partially ordered set) is a triple $(\mathbb{P}, \leq, \mathbb{1})$ such that $(\mathbb{P}, \leq)$ is a partial order and $\mathbb{1} \in \mathbb{P}$ is the largest element
- Elements of $\mathbb{P}$ are called forcing conditions
- $p \leq q$ is read "p extends q" or "p is stronger than q"

Partial orders

Definition

Let $\mathbb{P}$ be a poset. We say

- $p, q \in \mathbb{P}$ are compatible (written $p \not\perp q$) if they have a common extension, i.e. there is an $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$
- p, q are incompatible (written $p \perp q$) if they are not compatible

Partial orders

Definition

Let $\mathbb{P}$ be a poset.

- A antichain of $\mathbb{P}$ is a subset $A \subset \mathbb{P}$ whose elements are pairwise incompatible
- We say $\mathbb{P}$ has the countable chain condition (ccc) iff every antichain of $\mathbb{P}$ is countable

Partial orders

Example

Consider a subtree $\mathbb{P} \subset 2^{<\omega}$. Conditions in $\mathbb{P}$ are finite $\{0, 1\}$ sequences, and $p \leq q$ iff p is an end-extension of q.

- $p \not\perp q$ iff $p \leq q$ or vise versa
- $\mathbb{P}$ has the ccc, since $\mathbb{P}$ is at most countable

How about $\mathbb{P} \subset 2^{<\kappa}$ for some uncountable κ?

Partial orders

Example

For any I, J, let $\mathrm{Fn}(I, J)$ be the set of all finite partial functions from I to J, i.e. $p \in \mathrm{Fn}(I, J)$ iff p is finite, and p is a function with $\operatorname{dom} p \subset I$, $\operatorname{ran} p \subset J$

For $p, q \in \mathrm{Fn}(I, J)$, define $p \leq q$ iff $p \supset q$.

- $p \not\perp q$ iff they agree on $\operatorname{dom} p \cap \operatorname{dom} q$

Partial orders

Lemma

~~Fn~~ has the ccc if and only if $I = \emptyset$ or J is countable

$Fn(I, J)$

Partial orders

Definition

A family of sets $\mathcal{A}$ forms a delta system with root R iff for any $X, Y \in \mathcal{A}$ with $X \neq Y$, we have $X \cap Y = R$

Lemma

Let κ be an uncountable regular cardinal, and let $\mathcal{A}$ be a family of finite sets with card $A = \kappa$. Then there is a $\mathcal{B} \in [\mathcal{A}]^{\kappa}$ such that $\mathcal{B}$ forms a delta system.

$B \subseteq A$ and $|B| = \kappa$

each $X \in \mathcal{A}$ is finite, card $A = \kappa$, κ is uncountable regular,

Find a delta system $\mathcal{B} \subset \mathcal{A}$ with card $\mathcal{B} = \kappa$

Proof

Find $B_0 \in [A]^\kappa$ s.t. every $X \in B_0$ card $X = n$ for some fixed n.

We prove by induction on n.

If card $B_0 = \kappa$ and $\forall X \in B_0$ card $X = n$ then B_0 contains a delta system of cardinal κ

Case 1 $n = 1$, we have a delta system with $\emptyset$ root

Case 2 each $X \in B_0$, $|X| = n+1$

Consider $D_p = \{X \in B_0 \mid p \in X\}$ for $p \in \bigcup B_0$

Subcase 1

there is $p \in \bigcup B_0$ s.t. card $D_p = \kappa$

Then $D_p' = \{X / \{p\} \mid X \in D_p\}$ is a set of cardinality κ, and every element in it has cardinality n

By IH, there is a delta system D_p'' of cardinality κ s.t. $D_p'' \subseteq D_p'$

$D_p''' = \{X \cup \{p\} \mid X \in D_p''\}$ is a delta system $\subseteq A$

Subcase 2 every D_p has cardinality $< \kappa$

We construct a sequence of pairwise disjoint set of length κ

Given $\{X_\xi \mid \xi < \alpha\}$ built $(\alpha < \kappa)$

Let $Y = \bigcup_{\xi < \alpha} X_\xi$, Then card $Y < \kappa$

and card $\bigcup_{p \in Y} D_p < \kappa$ (since κ is regular)

Let X_α be the least in $B_0 / \bigcup_{p \in Y} D_p$

Then $X_\alpha \cap X_\xi = \emptyset$ for $\xi < \alpha$

So $\{X_\alpha \mid \alpha < \kappa\}$ is a delta system with $\emptyset$ root □

Assume $I \neq \emptyset$ and J is countable.

Show that $\mathrm{Fn}(I, J)$ has the ccc

Proof Assume to contradiction that there is an antichain $\mathcal{A} \subseteq \mathrm{Fn}(I,J)$ s.t. $|\mathcal{A}| = \aleph_1$.

Let $\mathcal{B} \subseteq \mathcal{A}$ s.t. $|\mathcal{B}| = \aleph_1$, and $\{\mathrm{dom}\, p \mid p \in \mathcal{B}\}$ is a delta system with root $S \subseteq I$

i.e. for $p, q \in \mathcal{B}$, $\mathrm{dom}\, p \cap \mathrm{dom}\, q = S$

Since $\mathcal{B}$ is an antichain

$\mathrm{card}\, \{p \restriction S \mid p \in \mathcal{B}\} = \aleph_1$.

But $\{p \restriction S \mid p \in \mathcal{B}\} \subseteq J^S$

and $|J^S| = \aleph_0$. ⊣⋈

Definition

Let $\mathbb{P}$ be a forcing poset. We say $D \subset \mathbb{P}$ is dense iff

$\forall p \in \mathbb{P} \exists q \in D \; q \leq p$

Partial orders

Example

Assume I is infinite and J is non-empty. Then sets of form $\left\{q \in \mathrm{Fn}(I, J) \;\middle|\; i \in \mathrm{dom}(q)\right\}$ and $\left\{q \in \mathrm{Fn}(I, J) \;\middle|\; j \in \mathrm{ran}(q)\right\}$ are dense whenever $i \in I$ and $j \in J$

Definition

Let $\mathbb{P}$ be a forcing poset. Then $F \subset \mathbb{P}$ is a filter on $\mathbb{P}$ iff

- $\mathbb{1} \in F$
- $\forall p, q \in F \exists r \in F\, (r \leq p \wedge r \leq q)$
- $\forall p, q \in \mathbb{P}\, (p \geq q \rightarrow q \in F \rightarrow p \in Q)$

F

Partial orders

Lemma

Let $\mathbb{P}$ be a forcing poset, $\mathcal{D}$ be a countable family of dense subsets of $\mathbb{P}$, and fix any $p \in \mathbb{P}$. Then there exists a filter G on $\mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$

In this case, we say G is a generic filter on $\mathbb{P}$ for $\mathcal{D}$

Intuitively, A generic filter G is what we want to build so that fulfills the requirements carried by those dense sets

Lemma

Let $\mathbb{P}$ be a forcing poset, $\mathcal{D}$ be a countable family of dense subsets of $\mathbb{P}$, and fix any $p \in \mathbb{P}$. Then there exists a filter G on $\mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$

In this case, we say G is a generic filter on $\mathbb{P}$ for $\mathcal{D}$

Intuitively, A generic filter G is what we want to build so that fulfills the requirements carried by those dense sets

Fine a generic filter G on $\mathbb{P}$ for $\mathcal{D}$

such that $p \in G$

Let $\mathcal{D} = \{ D_0, D_1, D_2 \cdot \}$

Let $p_0 \in D_0$ s.t. $p_0 \le p$

For n,

Let $p_{n+1} \in D_{n+1}$ s.t. $p_{n+1} \le p_n$.

Let $G = \{ q \mid \exists n \in \omega \; q \ge p_n \}$

Then G is a filter

i) $\mathbb{1} \in G$

ii) $p, q \in G$ s.t. $p \ge p_n$

$q \ge p_m$

Assume $m \ge n$, then $p \ge p_m$ and $p_m \in G$

$q \ge p_m$

iii) trivial

And $p \in G$

and G meet each D_n at p_n

Generic Extensions

From now on, we assume M is a c.t.m. for ZFC

Convention

When we say $\mathbb{P}$ is a forcing poset in M, we mean $(\mathbb{P}, \leq, \mathbb{1}) \in M$, and M think $(\mathbb{P}, \leq, \mathbb{1})$ is a forcing poset.

Note: Being a forcing poset is absolute for M

Generic Extensions

From now on, we assume M is a c.t.m. for ZFC

Convention

When we say $\mathbb{P}$ is a forcing poset in M, we mean $(\mathbb{P}, \leq, \mathbb{1}) \in M$, and M think $(\mathbb{P}, \leq, \mathbb{1})$ is a forcing poset.

Note: Being a forcing poset is absolute for M

Generic Extensions

Definition

Let $\mathbb{P}$ be a forcing poset in M. We say G is a $\mathbb{P}$-generic over M iff G is a filter on $\mathbb{P}$ and $G \cap D \neq \emptyset$ for all dense set $D \subset \mathbb{P}$ such that $D \in M$

Generic Extensions

Lemma (Generic Filter Existence)

Let $\mathbb{P}$ be a forcing poset in M. Then for every $p \in \mathbb{P}$, there exists a filter G on $\mathbb{P}$ such that $p \in G$ and G is $\mathbb{P}$-generic over M

Generic Extensions

Normally, we do not expect such generic $G \in M$

Lemma

If $\mathbb{P} \in M$ is atomless (namely, there is no $r \in \mathbb{P}$ such that for any $p, q \leq r$, $p \not\perp q$) and the filter G is $\mathbb{P}$-generic over M, then $G \notin M$

Next on Set Theory

- $\mathbb{P}$-names
- Generic Extension