# Set Theory II

# 集合论 II

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

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# Previously on Set Theory

- In ZF we showed: $(\mathrm{ZF} + V = L)^L$
- In $\mathrm{ZF} + V = L$ we proved: $\mathrm{AC}, \mathrm{GCH}$
    - To prove AC, we actually defined a global well-ordering of $L$
    - To show GCH, we used a "Skolem Hull"-like argument
- So we have $\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}(\mathrm{ZFC} + \mathrm{GCH})$

# Inner models

## Definition (Inner model)

We say a transitive class $M$ is an inner model if it is a " a model of ZF" and contains all ordinals, i.e. $\mathrm{OR} \subset M$.

Note it can be viewed as a definition in a metalanguage or a schema

Practically, when we say $M$ is an inner model, we mean $M$ satisfies a sufficient finite subset of ZF

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# Inner models

## Theorem

$L$ is the "smallest" inner model, i.e. If $M$ is an inner model, then $L \subset M$

# Inner models

EXE: Why this argument does not work for WF?

# Ordinal Definable Sets

Informally, *OD* is the class of all sets that are definable in $V$ with parameters from OR

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# Ordinal Definable Sets

## Definition

Let $M$ be a transitive set. Define $OD_M$ to be the set of all elements $a \in M$ that a definable in $(M, \in)$ with parameters in $\mathrm{OR} \cap M$, i.e.

$$OD_M = \Big\{ a \in M \,\Big|\, \exists \varphi(x, y_1, \ldots, y_n) \exists \alpha_1, \ldots, \alpha_n \in \mathrm{OR} \cap M \\ M \models \varphi(a, \vec{\alpha}) \wedge \forall x \big( M \models \varphi(x, \vec{\alpha}) \rightarrow x = a \big) \Big\}$$

Define $OD = \bigcup_{\alpha \in \mathrm{OR}} OD_{V_\alpha}$

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Define $OD = \bigcup_{\alpha \in \mathrm{OR}} OD_{V_\alpha}$

# Ordinal Definable Sets

## Lemma

Fix a formula $\varphi(x, y_1, \ldots, y_n)$, we can proof in ZF that

$$\forall \alpha_1, \ldots, \alpha \in \mathrm{OR} \forall a \Big[ \varphi(a, \vec{\alpha}) \wedge \forall x \Big( \varphi(x, \vec{\alpha}) \to x = a \Big) \to a \in OD \Big]$$

This lemma says "every ordinal definable set (in $V$) is in $OD$"

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# Ordinal Definable Sets

## Lemma

There is a formula $\psi(x, y)$ such that in ZF we can show

$$\forall\alpha \in \mathrm{OR}\exists! x\psi(x, \alpha) \wedge \forall x\Big[x \in OD \rightarrow \exists\alpha \in \mathrm{OR}\psi(x, \alpha)\Big]$$

It seems saying that "every set in $OD$ is actually definable by the formula $\psi$ with an ordinal as parameter"

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$$\forall \alpha \in \mathrm{OR} \exists ! x \psi(x, \alpha) \wedge \forall x \left[ x \in OD \to \exists \alpha \in \mathrm{OR} \psi(x, \alpha) \right]$$

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# Ordinal Definable Sets

## Fact

- For each $\alpha$, $V_\alpha \in OD$
- $OD$ is not transitive unless $V = OD$

# Ordinal Definable Sets

## Definition

We define

$$HOD = \left\{ x \in OD \,\middle|\, \mathrm{TC}(x) \subset OD \right\}$$

# Ordinal Definable Sets

## Fact

- $\mathrm{OR} \subset HOD \subset OD$
- $HOD$ is transitive
- ZFC hold in $HOD$

## Inner models

Can we prove $\mathrm{Con}(\mathrm{ZFC}) \to \mathrm{Con}(\mathrm{ZFC} + \neg\mathrm{CH})$ by defining an inner model?

# Forcing idea

The only possible way left is to build an “outer model” extending $V$

# Forcing idea

There are two ways to think about an outer model

- countable transitive model
- forcing language

# Forcing idea

Assuming $V_\kappa \vDash \mathrm{ZFC}$, then there is a countable transitive (set) model $M$ of ZFC by LST theorem and Mostowski Collapse

**Fact**

$\omega^M = \omega$

But $\omega_1^M, \omega_2^M, \ldots$ are actually countable ordinals, and there are only countable many reals in $M$

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## Forcing idea

Therefore it is possible to add $\aleph_2^M$ many (actually countable many) subsets of $\omega = \omega^M$ into the model to violate CH

Problem:

- We need also add all subsets “constructible from ” these reals
- We need to ensure that $\omega_2^M$ is still $\omega_2$ in the new model

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Problem:

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Let $M$ be c.t.m. of ZFC

Then Con(ZFC) hold in $V$, But it is a $\Pi^0_1$-sentence

So $M \models$ Con(ZFC). Now we have $M$ is a model of ZFC + Con(ZFC)

Then Con(ZFC + Con(ZFC)) hold in $V$, so is in $M$ ...

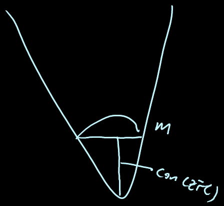
## Forcing idea

An even more troublesome problem:

Assuming there exists a c.t.m of ZFC is much stronger than Con(ZFC)

### Lemma

If there is a c.t.m of ZFC, then Con(ZFC), Con(ZFC + Con(ZFC)), etc.



# Forcing idea

On the other hand,, we will develop a so called forcing language.

The syntax is essentially the same with the normal language of set theory except that we introduce a class of "names" in our universe $V$, but with these names, we can "talk" about "possible" objects outside our universe

# Forcing idea

We use a set of partially ordered "conditions" to characterise the possibilities.

Recall: We say a set $\mathbb{P}$ is partially ordered by $\leq$, if $\leq \subset \mathbb{P} \times \mathbb{P}$ and $\leq$ is transitive and antisymmetry

We call $p, q \in \mathbb{P}$ conditions, and $p$ is stronger (containing more information) than $q$ if $p \leq q$

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# Forcing idea

The semantics of forcing language is forcing relation. We will define expressions like

$$p \Vdash \varphi(\dot{x})$$

reading "$p$ force $\varphi(\dot{x})$" (to be true), where $p$ is a condition, $\varphi$ is a formula in the forcing language and $\dot{x}$ is a name in our universe

# Forcing idea

The Intuition of $p \Vdash \varphi(\dot{x})$ is that if the condition $p$ is satisfied than $\varphi(x)$ holds in the outer model whatever $x$ is interpreted not violating $p$

We will show that every condition forces ZF, no condition force $0 = 1$ (assuming Con(ZF)), and in some case $\neg$CH is also forced

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# Next on Set Theory

- Facts on partial orders
- Generic extensions