Set Theory II

集合论 II

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Previously on Set Theory

- In ZF we showed: $(\mathrm{ZF} + V = L)^L$
- In $\mathrm{ZF} + V = L$ we proved: $\mathrm{AC}, \mathrm{GCH}$
  - To prove AC, we actually defined a global well-ordering of L
  - To show GCH, we used a "Skolem Hull"-like argument
- So we have $\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}(\mathrm{ZFC} + \mathrm{GCH})$

Inner models

Definition (Inner model)

We say a transitive class M is an inner model if it is a " a model of ZF" and contains all ordinals, i.e. $\mathrm{OR} \subset M$.

Note it can be viewed as a definition in a metalanguage or a schema

Practically, when we say M is an inner model, we mean M satisfies a sufficient finite subset of ZF

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Inner models

Theorem

L is the "smallest" inner model, i.e. If M is an inner model, then $L \subset M$

Inner models

EXE: Why this argument does not work for WF?

Ordinal Definable Sets

Informally, OD is the class of all sets that are definable in V with parameters from OR

But definability in V is not definable in V

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Ordinal Definable Sets

Definition

Let M be a transitive set. Define OD_M to be the set of all elements $a \in M$ that a definable in $(M, \in)$ with parameters in $\mathrm{OR} \cap M$, i.e.

$$OD_M = \left\{ a \in M \,\middle|\, \exists \varphi(x, y_1, \ldots, y_n) \exists \alpha_1, \ldots, \alpha_n \in \mathrm{OR} \cap M \right.$$
$$\left. M \vDash \varphi(a, \vec{\alpha}) \wedge \forall x \big(M \vDash \varphi(x, \vec{\alpha}) \to x = a \big) \right\}$$

Define $OD = \bigcup_{\alpha \in \mathrm{OR}} OD_{V_\alpha}$

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Define $OD = \bigcup_{\alpha \in \mathrm{OR}} OD_{V_\alpha}$

Ordinal Definable Sets

Lemma

Fix a formula $\varphi(x, y_1, \ldots, y_n)$, we can proof in ZF that

$$\forall \alpha_1, \ldots, \alpha \in \mathrm{OR} \forall a \Big[\varphi(a, \vec{\alpha}) \wedge \forall x \Big(\varphi(x, \vec{\alpha}) \to x = a \Big) \to a \in OD \Big]$$

This lemma says “every ordinal definable set (in V) is in OD”

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Ordinal Definable Sets

Lemma

There is a formula $\psi(x, y)$ such that in ZF we can show

$$\forall \alpha \in \mathrm{OR} \exists ! x \psi(x, \alpha) \wedge \forall x \Big[x \in OD \rightarrow \exists \alpha \in \mathrm{OR} \psi(x, \alpha) \Big]$$

It seems saying that "every set in OD is actually definable by the formula ψ with an ordinal as parameter"

Ordinal Definable Sets

Lemma

There is a formula $\psi(x, y)$ such that in ZF we can show

$$\forall \alpha \in \mathrm{OR} \exists ! x \psi(x, \alpha) \land \forall x \left[x \in OD \to \exists \alpha \in \mathrm{OR} \psi(x, \alpha) \right]$$

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Ordinal Definable Sets

Fact

- For each α, $V_\alpha \in OD$
- OD is not transitive unless $V = OD$

Ordinal Definable Sets

Definition

We define

$$HOD = \left\{ x \in OD \,\middle|\, \mathrm{TC}(x) \subset OD \right\}$$

Ordinal Definable Sets

Fact

- $\mathrm{OR} \subset HOD \subset OD$
- HOD is transitive
- ZFC hold in HOD

Inner models

Can we prove $\mathrm{Con}(\mathrm{ZFC}) \rightarrow \mathrm{Con}(\mathrm{ZFC} + \neg\mathrm{CH})$ by defining an inner model?

Forcing idea

The only possible way left is to build an "outer model" extending V

Forcing idea

There are two ways to think about an outer model

- countable transitive model
- forcing language

Forcing idea

Assuming $V_\kappa \vDash \mathrm{ZFC}$, then there is a countable transitive (set) model M of ZFC by LST theorem and Mostowski Collapse

Fact

$\omega^M = \omega$

But $\omega_1^M, \omega_2^M, \ldots$ are actually countable ordinals, and there are only countable many reals in M

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Therefore it is possible to add $\aleph_2^M$ many (actually countable many) subsets of $\omega = \omega^M$ into the model to violate CH

Problem:

- We need also add all subsets "constructible from " these reals
- We need to ensure that ω_2^M is still ω_2 in the new model

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Forcing idea

Let m be c.t.m. of ZFC

Then Con(ZFC) hold in V, But it is a Π^0_1-sentence

So $m \models$ Con(ZFC). Now we have m is a model of ZFC + Con(ZFC)

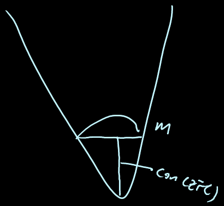
Then Con(ZFC + Con(ZFC)) hold in V, so is in m ...

An even more troublesome problem:

Assuming there exists a c.t.m of ZFC is much stronger than Con(ZFC)

Lemma

If there is a c.t.m of ZFC, then Con(ZFC), Con(ZFC + Con(ZFC)), etc.



Forcing idea

On the other hand,, we will develop a so called forcing language.

The syntax is essentially the same with the normal language of set theory except that we introduce a class of “names” in our universe V, but with these names, we can “talk” about “possible” objects outside our universe

Forcing idea

We use a set of partially ordered "conditions" to characterise the possibilities.

Recall: We say a set $\mathbb{P}$ is partially ordered by $\leq$, if $\leq \subset \mathbb{P} \times \mathbb{P}$ and $\leq$ is transitive and antisymmetry

We call $p, q \in \mathbb{P}$ conditions, and p is stronger (containing more information) than q if $p \leq q$

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Forcing idea

The semantics of forcing language is forcing relation. We will define expressions like

$$p \Vdash \varphi(\dot{x})$$

reading "p force $\varphi(\dot{x})$" (to be true), where p is a condition, φ is a formula in the forcing language and $\dot{x}$ is a name in our universe

Forcing idea

The Intuition of $p \Vdash \varphi(\dot{x})$ is that if the condition p is satisfied than $\varphi(x)$ holds in the outer model whatever x is interpreted not violating p

We will show that every condition forces ZF, no condition force $0 = 1$ (assuming Con(ZF)), and in some case $\neg$CH is also forced

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Next on Set Theory

- Facts on partial orders
- Generic extensions