

Set Theory II

集合论 II

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

Spring 2016

Previously on Set Theory

- Definition of L_α $L_{\beta+1} = \mathcal{D}^+(L_\beta)$
- Basic facts of L_α
- $L \models \text{ZF}$ $(\text{ZF})^L$
- L_α is absolute for transitive model of ZF – Pow, so $(V = L)^L$

Gödel's L

We show

- AC^L
- GCH^L , where **GCH** is the following statement

$$\forall \alpha \in \text{OR}(\text{card } 2^\alpha = \alpha^+)$$

Global choice in L

Recall: The following formulations of AC are equivalent

- Every family of mutually disjoint nonempty sets has a choice set / function
- Every set can be well-ordered
- Zorn's lemma

EXE: Proof that that are equivalent in ZF

Global choice in L

We define recursively a sequence of order relation \triangleleft_α on L_α , such that

- Each \triangleleft_α is a well-ordering of L_α
- For $\alpha < \beta$, \triangleleft_β is an **end-extension** of \triangleleft_α , which means $\triangleleft_\alpha \subset \triangleleft_\beta$, and for any $a \in L_\alpha$, $b \in L_\beta \setminus L_\alpha$, we have $a \triangleleft_\beta b$

Global choice in L

Fix $\alpha \in \text{OR}$

- In case α is limit, let $\triangleleft_\alpha = \bigcup_{\gamma < \alpha} \triangleleft_\gamma$
- In case $\alpha = \beta + 1$,
 - For subcase $a, b \in L_\beta$. Let $a \triangleleft_\alpha b$ if $a \triangleleft_\beta b$
 - For subcase $a \in L_\beta, b \in L_{\beta+1} \setminus L_\beta$. Let $a \triangleleft_\alpha b$

Global choice in L

Fix $\alpha \in \text{OR}$

- In case $\alpha = \beta + 1$
 - For subcase $a, b \in L_{\beta+1} \setminus L_\beta$.

Fix an enumeration of all set theory formulas $\{\varphi_i \mid i < \omega\}$
and an well-ordering \triangleleft_β^n of L_β^n for each n .

Let i, j be the least such that $a = \{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{p})\}$ and
 $b = \{x \in L_\beta \mid \varphi_j^{L_\beta}(x, \vec{q})\}$ for some $\vec{p}, \vec{q} \in L_\beta^{<\omega}$ respectively

- For the subsubcase that $i < j$, let $a \triangleleft_\alpha b$

Global choice in L

Fix $\alpha \in \text{OR}$

- In case $\alpha = \beta + 1$
 - For subcase $a, b \in L_{\beta+1} \setminus L_\beta$.
 - For the subsubcase that $i = j$, let $\vec{p}, \vec{q} \in L_\beta^n$ be the least such that $a = \{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{p})\}$ and $b = \{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{q})\}$.
Let $a \triangleleft_\alpha b$ if $\vec{p} \triangleleft_\beta^n \vec{q}$

Note that if $i = j$ and $\vec{p} = \vec{q}$, then $a = b$

Global choice in L

Fix $\alpha \in \text{OR}$

- In case $\alpha = \beta + 1$
 - For subcase $a, b \in L_{\beta+1} \setminus L_\beta$.
 - For the subsubcase that $i = j$, let $\vec{p}, \vec{q} \in L_\beta^n$ be the least such that $a = \{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{p})\}$ and $b = \{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{q})\}$.
Let $a \triangleleft_\alpha b$ if $\vec{p} \triangleleft_\beta^n \vec{q}$
Note that if $i = j$ and $\vec{p} = \vec{q}$, then $a = b$

Global choice in L

Lemma

Each L_α is well-ordered by \triangleleft_α , and \triangleleft (defined as $a \triangleleft b$ iff there exists $\alpha \in \text{OR}$ such that $a \triangleleft_\alpha b$) is a global well-ordering of L

Global choice in L

Corollary

$ZF \vdash (\triangleleft \text{ is a global well-ordering of the universe })^L$. In particular, $ZF \vdash AC^L$

GCH in L

Definition

Let M be a transitive set, define $\text{Ord}^M = M \cap \text{OR} =$ the set of ordinals in M

Fact Ord^M is an ordinal

ω , If M is transitive and $M \models \text{ZF-Pow}$, Then

Ord^M is limit

Lemma

Let M be a transitive set such that $M \models \text{ZF-Pow}$. Then

$$M \models V=L$$

$$\text{i.e. } (V=L)^M$$

$$M \models V=L \iff M = L_{\text{Ord}^M}$$

$$\text{i.e. } \forall x \in M \exists \beta \in \text{Ord}^M \underbrace{(x \in L_\beta)^M}_{x \in L_\beta}$$

Show

1 $M \subset L_{\alpha(M)}$

2 $L_{\alpha(M)} \subset M$

we) Fix $x \in M$

Since $m \neq \emptyset \Rightarrow L$ is.

$$\forall \alpha \in M \exists \beta \quad x \in (L_{\beta})^M = L_{\beta}$$

$\forall \alpha$ there is β s.t. $x \in L_{\beta+1}$

Let β be the least such.

i.e. $x \notin L_{\beta} \hookrightarrow$ since rank is absolute for M

$$(\text{rank}(x))^M = \beta, \text{ therefore } \beta \in M$$

"

$$\text{rank}(x)$$

$$\text{i.e. } \beta \in M \cap \text{ord} = \text{ord}(M)$$

$\alpha \rightarrow L_{\alpha}$ is absolute for M

i.e. For each $\alpha \in M \cap \text{ord}$,

we have $L_{\alpha} \in M$, so $L_{\alpha} \in M$

$$\text{i.e. } L_{\text{ord}(M)} = \bigcup_{\alpha \in \text{ord}(M)} L_{\alpha} \subseteq M$$

GCH in L

Lemma

If κ is a **regular uncountable cardinal**, then

$$L_\kappa \models \text{ZF} - \text{Pow} + V = L$$

We show:

$L_k \models$ Replacement

Fix $A \in L_k$.

Fix formula $\varphi(x, y, z_1, \dots, z_n)$ s.t.

$\forall z_1, \dots, z_n \in L_k \forall x \in A \exists! y \in L_k \varphi(x, y, z_1, \dots, z_n)$

By Replacement, there is a set B s.t.

$\forall x \in A \exists! y \in B \dots$

and $\forall y \in B \exists x \in A \varphi(x, y, \dots)$

For all $x \in A$, define $f_x = \text{rank}_L(y)$

where y is the unique s.t. $\varphi(x, y, \dots)$

Constr. $\{f_x \mid x \in A\}$, since each $f_x < k$

and $|A| \leq |L_{\text{rank}(A)}| = |\text{rank}(A)| < k$

Therefore $\sup_{x \in A} f_x = \beta < k$
for some β

i.e. $B \subset L_\beta$

Then $B \in L_\beta \subset L_k$

Since

$B = \{y \mid (\exists x \in A \varphi(x, y, z_1, \dots, z_n))\}^{L_k, y}$

and we can find $\theta < k$ s.t.

$L_\theta \prec_\varphi L_k$

By a skolem argument
Hull

GCH in L

Theorem ($V = L$)

For all cardinal $\kappa \geq \omega$, $L_\kappa = H(\kappa)$ (the family of all sets whose transitive closure has cardinality $< \kappa$). Hence, GCH holds

- $L_\kappa = H(\kappa)$ implies GCH

Fix λ , then $P(\omega) \subseteq H(\omega^+) = L_{\lambda^+}$

and $|L_{\lambda^+}| = \lambda^+$

$\therefore |P(\omega)| = 2^\omega \leq \lambda^+$

- prove $L_\kappa = H(\kappa)$ by

induction on cardinal $\kappa \geq \omega$

For κ is limit cardinal

$x \in L_\kappa \Leftrightarrow x \in L_\beta$ for sum $\beta < \kappa$

$\Rightarrow x \in L_{\beta^+}$

$\Rightarrow x \in H(\beta^+) \subseteq H(\kappa)$

since $\beta^+ < \kappa$

$x \in H(\kappa) \Rightarrow x \in H(\omega)$ for some $\lambda < \kappa$

$\Rightarrow x \in L_\lambda \in L_\kappa$

\uparrow
 \cup

For $\kappa = \lambda^+$

i.e. κ is uncountable regular cardinal.

and $L_\kappa \models \exists T - Pow + V=L$

Fix $x \in L_\kappa$, then $x \in L_\beta$ ($\beta < \kappa$)

so $tc(x) \subseteq L_\beta$, and $|L_\beta| < \kappa$

GCH in L

Corollary

GCH^L

Corollary

$$\text{Con}(\text{ZF}) \rightarrow \text{Con}(\text{ZFC} + \text{GCH})$$

Note that this can be a theorem in finitistic mathematics

Corollary

$$\text{Con}(\text{ZF}) \rightarrow \text{Con}(\text{ZFC} + \text{GCH})$$

Note that this can be a theorem in finitistic mathematics

Next on Set Theory

- More on L
- Introduction of Forcing