

Set Theory II

集合论 II

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Previously on Set Theory

- Definition of L_α $L_{\beta+1} = \mathcal{D}^+(L_\beta)$
- Basic facts of L_α
- $L \models \text{ZF}$ $(\text{ZF})^L$
- L_α is absolute for transitive model of ZF – Pow, so $(V = L)^L$

Gödel's L

We show

- AC^L
- GCH^L , where **GCH** is the following statement

$$\forall \alpha \in \text{OR}(\text{card } 2^\alpha = \alpha^+)$$

Global choice in L

Recall: The following formulations of AC are equivalent

- Every family of mutually disjoint nonempty sets has a choice set / function
- Every set can be well-ordered
- Zorn's lemma

EXE: Proof that that are equivalent in ZF

Global choice in L

We define recursively a sequence of order relation \triangleleft_α on L_α , such that

- Each \triangleleft_α is a well-ordering of L_α
- For $\alpha < \beta$, \triangleleft_β is an **end-extension** of \triangleleft_α , which means $\triangleleft_\alpha \subset \triangleleft_\beta$, and for any $a \in L_\alpha$, $b \in L_\beta \setminus L_\alpha$, we have $a \triangleleft_\beta b$

Global choice in L

Fix $\alpha \in \text{OR}$

- In case α is limit, let $\triangleleft_\alpha = \bigcup_{\gamma < \alpha} \triangleleft_\gamma$
- In case $\alpha = \beta + 1$,
 - For subcase $a, b \in L_\beta$. Let $a \triangleleft_\alpha b$ if $a \triangleleft_\beta b$
 - For subcase $a \in L_\beta, b \in L_{\beta+1} \setminus L_\beta$. Let $a \triangleleft_\alpha b$

Global choice in L

Fix $\alpha \in \text{OR}$

- In case $\alpha = \beta + 1$
 - For subcase $a, b \in L_{\beta+1} \setminus L_\beta$.

Fix an enumeration of all set theory formulas $\{\varphi_i \mid i < \omega\}$
and an well-ordering \triangleleft_β^n of L_β^n for each n .

Let i, j be the least such that $a = \{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{p})\}$ and
 $b = \{x \in L_\beta \mid \varphi_j^{L_\beta}(x, \vec{q})\}$ for some $\vec{p}, \vec{q} \in L_\beta^{<\omega}$ respectively

- For the subsubcase that $i < j$, let $a \triangleleft_\alpha b$

Global choice in L

Fix $\alpha \in \text{OR}$

- In case $\alpha = \beta + 1$
 - For subcase $a, b \in L_{\beta+1} \setminus L_\beta$.
 - For the subsubcase that $i = j$, let $\vec{p}, \vec{q} \in L_\beta^n$ be the least such that $a = \{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{p})\}$ and $b = \{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{q})\}$.
Let $a \triangleleft_\alpha b$ if $\vec{p} \triangleleft_\beta^n \vec{q}$

Note that if $i = j$ and $\vec{p} = \vec{q}$, then $a = b$

Global choice in L

Fix $\alpha \in \text{OR}$

- In case $\alpha = \beta + 1$
 - For subcase $a, b \in L_{\beta+1} \setminus L_\beta$.
 - For the subsubcase that $i = j$, let $\vec{p}, \vec{q} \in L_\beta^n$ be the least such that $a = \{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{p})\}$ and $b = \{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{q})\}$.
Let $a \triangleleft_\alpha b$ if $\vec{p} \triangleleft_\beta^n \vec{q}$
Note that if $i = j$ and $\vec{p} = \vec{q}$, then $a = b$

Global choice in L

Lemma

Each L_α is well-ordered by \triangleleft_α , and \triangleleft (defined as $a \triangleleft b$ iff there exists $\alpha \in \text{OR}$ such that $a \triangleleft_\alpha b$) is a global well-ordering of L

Global choice in L

Corollary

$ZF \vdash (\triangleleft \text{ is a global well-ordering of the universe })^L$. In particular, $ZF \vdash AC^L$

GCH in L

Definition

Let M be a transitive set, define $o(M) = M \cap \text{OR} =$ the set of ordinals in M

Fact $o(M)$ is an ordinal

ω , If M is transitive and $M \models \text{ZF-Pow}$, Then

$o(M)$ is limit

Lemma

Let M be a transitive set such that $M \models \text{ZF-Pow}$. Then

$$M \models V=L$$

$$\text{i.e. } (V=L)^M$$

$$M \models V=L \iff M = L_{o(M)}$$

$$\text{i.e. } \forall x \in M \exists \beta \in \text{Ord}_M \underbrace{(x \in L_\beta)^M}_{x \in L_\beta}$$

Show

1 $M \subset L_{\alpha}(M)$

2 $L_{\alpha}(M) \subset M$

we fix $x \in M$

Since $m \notin V \Rightarrow L$ is.

$$\forall x \in M \exists \beta \quad x \in (L_{\beta})^M = L_{\beta}$$

∴ there is β s.t. $x \in L_{\beta+1}$

Let β be the least such.

i.e. $x \notin L_{\beta}$

$$(\text{rank}(x))^M = \beta, \text{ therefore}$$

$$\text{rank}(x)$$

GCH in L

Lemma

If κ is a **regular uncountable cardinal**, then

$$L_\kappa \models \text{ZF} - \text{Pow} + V = L$$

We show:

$L_x \vDash \text{Replacement}$

GCH in L

Theorem ($V = L$)

For all cardinal $\kappa \geq \omega$, $L_\kappa = H(\kappa)$ (the family of all sets whose transitive closure has cardinality $< \kappa$). Hence, GCH holds

- $L_\kappa = H(\kappa)$ implies GCH

- prove $L_\kappa = H(\kappa)$ by
induction on cardinal $\kappa \geq \omega$

GCH in L

Corollary

GCH^L

Corollary

$$\text{Con}(\text{ZF}) \rightarrow \text{Con}(\text{ZFC} + \text{GCH})$$

Note that this can be a theorem in finitistic mathematics

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$$\text{Con}(\text{ZF}) \rightarrow \text{Con}(\text{ZFC} + \text{GCH})$$

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Next on Set Theory

- More on L
- Introduction of Forcing