Set Theory II

集合论 II

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Previously on Set Theory

- Definition of L_α $\quad L_{\beta+1} = \mathcal{D}^+(L_\beta)$
- Basic facts of L_α
- $L \models \mathrm{ZF}$ $\quad (\mathrm{ZF})^L$
- L_α is absolute for transitive model of ZF – Pow, so $(V = L)^L$

Gödel's L

We show

- AC^L
- GCH^L, where GCH is the following statement

$$\forall\alpha \in \mathrm{OR}(\mathrm{card}\, 2^{\alpha} = \alpha^{+})$$

Global choice in L

Recall: The following formulations of AC are equivalent

- Every family of mutually disjoint nonempty sets has a choice set / function
- Every set can be well-ordered
- Zorn's lemma

EXE: Proof that that are equivalent in ZF

Global choice in L

We define recursively a sequence of order relation $\lhd_\alpha$ on L_α, such that

- Each $\lhd_\alpha$ is a well-ordering of L_α
- For $\alpha < \beta$, $\lhd_\beta$ is an end-extension of $\lhd_\alpha$, which means $\lhd_\alpha \subset \lhd_\beta$, and for any $a \in L_\alpha, b \in L_\beta \setminus L_\alpha$, we have $a \lhd_\beta b$

Global choice in L

Fix $\alpha \in \mathrm{OR}$

- In case α is limit, let $\triangleleft_\alpha = \bigcup_{\gamma<\alpha}$ ~~L_γ~~ $\triangleleft_\gamma$
- In case $\alpha = \beta + 1$,
  - For subcase $a, b \in L_\beta$. Let $a \triangleleft_\alpha b$ if $a \triangleleft_\beta b$
  - For subcase $a \in L_\beta, b \in L_{\beta+1} \setminus L_\beta$. Let $a \triangleleft_\alpha b$

Global choice in L

Fix $\alpha \in \mathrm{OR}$

- In case $\alpha = \beta + 1$
  - For subcase $a, b \in L_{\beta+1} \setminus L_\beta$.

    Fix an enumeration of all set theory formulas $\{\varphi_i \mid i < \omega\}$ and an well-ordering $\lhd^n_\beta$ of L^n_β for each n.

    Let i, j be the least such that $a = \{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{p})\}$ and $b = \{x \in L_\beta \mid \varphi_j^{L_\beta}(x, \vec{q})\}$ for some $\vec{p}, \vec{q} \in L_\beta^{<\omega}$ respectively
    - For the subsubcase that $i < j$, let $a \lhd_\alpha b$

Global choice in L

Fix $\alpha \in \mathrm{OR}$

- In case $\alpha = \beta + 1$
  - For subcase $a, b \in L_{\beta+1} \setminus L_\beta$.
    - For the subsubcase that $i = j$, let $\vec{p}, \vec{q} \in L_\beta^n$ be the least such that $a = \left\{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{p})\right\}$ and $b = \left\{x \in L_\beta \mid \varphi_i^{L_\beta}(x, \vec{q})\right\}$.

      Let $a \lhd_\alpha b$ if $\vec{p} \lhd_\beta^n \vec{q}$

      Note that if $i = j$ and $\vec{p} = \vec{q}$, then $a = b$

Global choice in L

Fix $\alpha \in \mathrm{OR}$

- In case $\alpha = \beta + 1$
  - For subcase $a, b \in L_{\beta+1} \setminus L_\beta$.
    - For the subsubcase that $i = j$, let $\vec{p}, \vec{q} \in L_\beta^n$ be the least such that $a = \left\{x \in L_\beta \,\middle|\, \varphi_i^{L_\beta}(x, \vec{p})\right\}$ and $b = \left\{x \in L_\beta \,\middle|\, \varphi_i^{L_\beta}(x, \vec{q})\right\}$.

      Let $a \lhd_\alpha b$ if $\vec{p} \lhd_\beta^n \vec{q}$

      Note that if $i = j$ and $\vec{p} = \vec{q}$, then $a = b$

Global choice in L

Lemma

Each L_α is well-ordered by $\lhd_\alpha$, and $\lhd$ (defined as $a \lhd b$ iff there exists $\alpha \in \mathrm{OR}$ such that $a \lhd_\alpha b$) is a global well-ordering of L

Global choice in L

Corollary

ZF $\vdash$ ($\lhd$ is a global well-ordering of the universe $)^L$. In particular, ZF $\vdash$ AC^L

GCH in L

Definition

Let M be a transitive set, define $o(M) = M \cap \mathrm{OR}=$ the set of ordinals in M

Fact (1) $o(M)$ is an ordinal
(2) If M is transitive and $M \models ZF-Pow$, Then $o(M)$ is limit

Lemma

Let M be a transitive set such that $M \models \mathrm{ZF} - \mathrm{Pow}$. Then

$M \models V = L$
i.e. $(V=L)^M$
i.e. $\forall x \in M \exists \beta \in \mathrm{Ord} \cap M \ (x \in L_\beta)^M$
$x \in L_\beta$

$$M \models V = L \quad \Leftrightarrow \quad M = L_{o(M)}$$

Show

1. $M \subset L_{o(M)}$
2. $L_{o(M)} \subset M$

(1) Fix $x \in M$

Since $M \models V = L$ i.e.

$\forall x \in M \ \exists \beta \ x \in (L_\beta)^M = L_\beta$

i.e. there is β s.t. $x \in L_{\beta+1}$

let β be the least such.

i.e. $x \notin L_\beta$ & since rank is absolute for M

$(\text{rank}(x))^M = \beta$, therefore $\beta \in M$

$\text{rank}(x)$

i.e. $\beta \in M \cap \text{Ord} = o(M)$

$\alpha \to L_\alpha$ is absolute for M

i.e. for each $\alpha \in M \cap \text{Ord}$,

we have $L_\alpha \in M$, so $L_\alpha \subseteq M$

i.e. $L_{o(M)} = \bigcup_{\alpha < o(M)} L_\alpha \subseteq M$

GCH in L

Lemma

If κ is a regular uncountable cardinal, then

$$L_\kappa \models \mathrm{ZF} - \mathrm{Pow} + V = L$$

We show:

$L_\kappa \models$ Replacement

Fix $A \in L_\kappa$.

Fix formula $\varphi(x, y, z_1 \ldots z_n)$ s.t.

$\forall z_1 \ldots z_n \in L_\kappa \; \forall x \in A \; \exists! y \in L_\kappa \; \varphi(x, y, z_1 \ldots z_n)$

By Replacement, there is a set B s.t.

$\forall x \in A \; \exists! y \in B \ldots$

and $\forall y \in B \; \exists x \in A \; \varphi(x, y \ldots)$

For all $x \in A$, define $\beta_x = \mathrm{rank}_L(y)$

where y is the unique s.t. $\varphi(x, y \ldots)$

Consider $\{\beta_x \mid x \in A\}$, since each $\beta_x < \kappa$

and $|A| \leq |L_{\mathrm{rank}(A)}| = |\mathrm{rank}(A)| < \kappa$

Therefore $\sup_{x \in A} \beta_x = \beta < \kappa$ for some β

i.e. $B \subset L_\beta$

Then $B \in L_\gamma \subseteq L_\kappa$

Since

$B = \{y \mid (\exists x \in A \; \varphi(x, y, z_1 \ldots z_n))^{L_\kappa}\}$

and we can find $\theta < \kappa$ s.t.

$L_\theta \prec_\varphi L_\kappa$

By a skolem argument (Hull)

GCH in L

Theorem ($V = L$)

For all cardinal $\kappa \geq \omega$, $L_\kappa = H(\kappa)$ (the family of all sets whose transitive closure has cardinality $< \kappa$). Hence, GCH holds

- $L_\kappa = H(\kappa)$ implies GCH

Fix λ, then $P(\lambda) \subseteq H(\lambda^+) = L_{\lambda^+}$

and $|L_{\lambda^+}| = \lambda^+$

i.e. $|P(\lambda)| = 2^\lambda \leq \lambda^+$

- prove $L_\kappa = H(\kappa)$ by induction on cardinal $\kappa \geq \omega$

For κ is limit cardinal

$x \in L_\kappa \iff x \in L_\beta$ for some $\beta < \kappa$

$\Rightarrow x \in L_{\beta^+}$

$\Rightarrow x \in H(\beta^+) \subseteq H(\kappa)$

since $\beta^+ < \kappa$

$x \in H(\kappa) \Rightarrow x \in H(\lambda)$ for some $\lambda < \kappa$

$\Rightarrow x \in L_\lambda \subseteq L_\kappa$

IH

For $\kappa = \lambda^+$

i.e. κ is uncountable regular cardinal.

and $L_\kappa \models ZF - Pow + V = L$

Fix $x \in L_\kappa$, then $x \in L_\beta$ $(\beta < \kappa)$

so $tc(x) \subseteq L_\beta$, and $|L_\beta| < \kappa$

so $L_\kappa \subseteq H(\kappa)$.

Fix $x \in H(\kappa)$,

Since $V = L$, we have $x \in L_\theta$ for some θ

Let $T = tc(x)$ then $|T| < \kappa$

By L-S theorem, there is $M \prec L_\theta$

s.t. $T \subseteq M$, and $|M| = |T|$

Let M' be the transitive collapse of M,

Then $M' \models ZF - Pow + V = L$ s.t.

$M' = L_\beta$ for som β, and $|\beta| = |L_\beta| = |M| = |T| < \kappa$

so $x \in T \subseteq M' = L_\beta \in L_\kappa$

GCH in L

Corollary

GCH^L

Corollary

$\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}(\mathrm{ZFC} + \mathrm{GCH})$

Note that this can be a theorem in finitistic mathematics

Corollary

$\text{Con(ZF)} \to \text{Con(ZFC} + \text{GCH)}$

Note that this can be a theorem in finitistic mathematics

Next on Set Theory

- More on L
- Introduction of Forcing