Set Theory II

集合论 II

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Previously on Set Theory

- Δ_0 (module logical equivalence) formulas are absolute for transitive classes (no additional precondition)
- Δ_0 (module Λ) formulas are absolute for transitive classes satisfying Λ. It is also true for Δ_1 formulas
- Δ_0 or Δ_1 in parameters are absolute if the parameters are
- Recursive definitions preserve absoluteness
- Functions are absolute if it is still a function and absolute as a relation

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Constructible sets

Recall:

Informally, we define $\mathcal{D}(A, P)$ ($P \subset A$)to be the collection of all subsets X of A which are definable in $(A, \in)$ from parameters from P, i.e. there exists a formula $\varphi(x, z_1, \ldots, z_n)$ and $p_1, \ldots, p_n \in P$ such that

$$X = \left\{ a \in A \,\middle|\, (A, \in) \vDash \varphi[a, p_1, \ldots, p_n] \right\}$$

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Constructible sets

Formally, we define

$$\mathcal{D}(A,P) = \Big\{ X \subset A \;\Big|\; \exists n, k \in \omega \exists s \in P^k \Big(n \text{ is a } \mathcal{L}_\in \text{ formula with at most } k+1 \text{ free variables} \wedge X \text{ is defined by } n \text{ with parameters } s \text{ in } (A, \in) \Big) \Big\}$$

$$\mathcal{D}^+(A) = \mathcal{D}(A, A)$$

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Lemma (meta-theorem)

For each set theory formula $\varphi(x, z_1, \ldots, z_n)$,

$$\mathrm{ZF}^- - \mathrm{Pow} \vdash \forall A \forall p_1, \ldots, p_n \in A\Big[\big\{a \in A \;\big|\; \varphi^A(a, p_1, \ldots, p_n)\big\} \in \mathcal{D}^+(A)\Big]$$

Constructible sets

Fact

Every finite subset of A is in $\mathcal{D}^+(A)$

Constructible sets

Definition

- $L_0 = \emptyset$
- $L_{\beta+1} = \mathcal{D}^+(L_\beta)$
- $L_\gamma = \bigcup_{\alpha<\gamma} L_\alpha$ for limit ordinal γ
- $L = \bigcup_{\alpha\in \mathrm{OR}} L_\alpha$

$V_0 = \emptyset$

$V_{\beta+1} = P(V_\beta)$

⋮

Constructible sets

Lemma

- $L_\alpha \subset V_\alpha$
- $L_\beta \subset L_{\beta+1}$
- L_α is transitive
- $\alpha \leq \beta \rightarrow L_\alpha \subset L_\beta$

Constructible sets

Induction on β

For limit case

$L_\beta \cap OR = \bigcup_{\alpha\in\beta} L_\alpha \cap OR$

$= \bigcup_{\alpha\in\beta} (L_\alpha \cap OR)$

$= \bigcup_{\alpha\in\beta} \alpha$ ← by induction hypothesis

$= \beta$

For $\beta = \gamma+1$

$L_{\gamma+1} \cap OR = L_\gamma \cup D^+(L_\gamma) \cap OR$

$= \gamma \cup (D^+(L_\gamma) \cap OR)$

L_γ

$\{\gamma\}$

$L_{\gamma+1} \cap OR \subseteq V_{\gamma+1} \cap OR = \gamma+1$

Lemma

$L_\beta \cap \mathrm{OR} = \beta$ for any ordinal β

It is sufficient to show $\gamma \in D^+(L_\gamma)$

Note: $\gamma = \{x \in L_\gamma \mid x \text{ is ordinal}\} = \{x \in L_\alpha \mid (x \text{ is ordinal})^{L_\gamma}\}$

We want to show OR is absolute for L_γ

Δ_0

$L_\gamma \models x \text{ is ord}$

$\varphi(x)$

Constructible sets

Definition

For $x \in L$, we define the *L-rank* of x,

$$\mathrm{rank}_L(x) = \min\left\{\alpha \in \mathrm{OR} \,\middle|\, x \in L_{\alpha+1}\right\}$$

Constructible sets

Fact

- $L_\alpha = \left\{ x \in L \,\middle|\, \mathrm{rank}_L(x) < \alpha \right\}$
- $L_{\alpha+1} \setminus L_\alpha = \left\{ x \in L \,\middle|\, \mathrm{rank}_L(x) = \alpha \right\}$
- $\mathrm{rank}_L\big(L(\alpha)\big) = \mathrm{rank}_L(\alpha) = \alpha$

Constructible sets

Fact

- Every finite subset of L_α is in $L_{\alpha+1}$
- $L_\alpha = V_\alpha$ for $\alpha \leq \omega$
- (AC) $L_{\omega+1} \subsetneq V_{\omega+1} = P(V_\omega)$ $|V_{\omega+1}| > |V_\omega| = \omega$

$|L_{\omega+1}| = \omega \times \omega^{<\omega} = \omega$

Theorem ((meta-theorem))

All axioms of ZF holds in L

Lemma

Suppose that M is a transitive class and Comprehension Schema holds in M, and assume that for every subset $x \subset M$, there is a set $y \in M$ such that $x \subset y$. Then All axioms of ZF holds in M

Constructible sets

Theorem ((meta-theorem))

All axioms of ZF holds in L

Lemma

Separation

Suppose that M is a transitive class and Comprehension Schema holds in M, and assume that for every subset $x \subset M$, there is a set $y \in M$ such that $x \subset y$. Then All axioms of ZF holds in M

Theorem ((meta-theorem))

All axioms of ZF holds in L

Lemma

Suppose that M is a transitive class and Comprehension Schema holds in M, and assume that for every subset $x \subset M$, there is a set $y \in M$ such that $x \subset y$. Then All axioms of ZF holds in M

(1) ■ Comprehension holds in L

(2) ■ for each $x \in L$, there is a $y \in L$ such that $x \subset y$

(2) Fix $x \in L$. Let $\alpha = \mathrm{rank}_L(x)$

Then $x \in L_{\alpha+1}$ i.e.

$x \subseteq L_\alpha$ and $L_\alpha \in L_{\alpha+1} \subseteq L$

(1) Fix $X \in L$, $\varphi(x, z_1 \cdots z_n)$, $p_1 \cdots p_n \in L$

We want to show

$$A = \{x \in X \mid \varphi^L(x, p_1 \cdots p_n)\} \in L$$

It is sufficient to find some β (large enough) s.t.

$$\varphi^L(x, p_1 \cdots p_n) \leftrightarrow \varphi^{L_\beta}(x, p_1 \cdots p_n) \quad \text{for } x \in X$$

Let Σ_0 be the set of all sub-formulas of $\varphi(x, y_1 \cdots y_n)$

(Assume φ contains only $\exists$)

⋮

Let L_β be the "Skolem Hull" for φ of L i.e.

$$L_\beta \prec_\varphi L$$

so,

$$A = \{x \in X \mid \varphi^{L_\beta}(x, p_1 \cdots p_n)\} \in L_{\beta+1} \subseteq L$$

Constructible sets

Lemma

The function $\alpha \to L_\alpha$ is absolute for any transitive class $M \models \mathrm{ZF} - \mathrm{Pow}$

Corollary

$(V = L)^L$

Constructible sets

Lemma $L \cap OR = OR$

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The function $\alpha \to L_\alpha$ is absolute for any transitive class $M \models \mathrm{ZF} - \mathrm{Pow}$

Corollary

$(V = L)^L$

$(\forall x \exists \alpha \; x \in L_\alpha)^L = \forall x \in L \; \exists \alpha \; x \in L_\alpha^L$

L_α
$\shortparallel$
L

- L_α is well-defined in transitive $M \models \mathrm{ZF} - \mathrm{Pow}$
- $L_\alpha^M = L_\alpha$

Next on Set Theory

- $V = L \models \mathrm{AC}$
- $V = L \models \mathrm{GCH}$