

# Set Theory II

## 集合论 II

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# Previously on Set Theory

- $\Delta_0$  (module logical equivalence) formulas are absolute for transitive classes (no additional precondition)
- $\Delta_0$  (module  $\wedge$ ) formulas are absolute for transitive classes satisfying  $\wedge$ . It is also true for  $\Delta_1$  formulas
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- Recursive definitions preserve absoluteness
- **Functions** are absolute if it is still a function and absolute as a relation

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# Constructible sets

Recall:

**Informally**, we define  $\mathcal{D}(A, P)$  ( $P \subset A$ ) to be the collection of all subsets  $X$  of  $A$  which are definable in  $(A, \in)$  from parameters from  $P$ , i.e. there exists a formula  $\varphi(x, z_1, \dots, z_n)$  and  $p_1, \dots, p_n \in P$  such that

$$X = \{a \in A \mid (A, \in) \models \varphi[a, p_1, \dots, p_n]\}$$

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# Constructible sets

Formally, we define

$$\mathcal{D}(A, P) = \left\{ X \subset A \mid \exists n, k \in \omega \exists s \in P^k \left( n \text{ is a } \mathcal{L}_\epsilon \text{ formula with} \right. \right. \\ \left. \left. \begin{array}{l} \text{at most } k + 1 \text{ free variables} \\ \wedge X \text{ is defined by } n \text{ with parameters } s \text{ in } (A, \epsilon) \right) \right\}$$

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# Constructible sets

Lemma (meta-theorem)

For each set theory formula  $\varphi(x, z_1, \dots, z_n)$ ,

$$\text{ZF}^- - \text{Pow} \vdash \forall A \forall p_1, \dots, p_n \in A \left[ \begin{array}{l} \{a \in A \mid \varphi^A(a, p_1, \dots, p_n)\} \in \mathcal{D}^+(A) \end{array} \right]$$

# Constructible sets

Fact

Every finite subset of  $A$  is in  $\mathcal{D}^+(A)$

# Constructible sets

## Definition

- $L_0 = \emptyset$
- $L_{\beta+1} = \mathcal{D}^+(L_\beta)$
- $L_\gamma = \bigcup_{\alpha < \gamma} L_\alpha$  for limit ordinal  $\gamma$
- $L = \bigcup_{\alpha \in \text{OR}} L_\alpha$

$$V_0 = \emptyset$$

$$V_{\beta+1} = \mathcal{P}(V_\beta)$$

⋮



# Constructible sets

## Lemma

- $L_\alpha \subset V_\alpha$
- $L_\beta \subset L_{\beta+1}$
- $L_\alpha$  is transitive
- $\alpha \leq \beta \rightarrow L_\alpha \subset L_\beta$

# Constructible sets

Induction on  $\beta$

For limit case

$$\begin{aligned} L_\beta \cap OR &= \bigcup_{\alpha < \beta} L_\alpha \cap OR \\ &= \bigcup_{\alpha < \beta} (L_\alpha \cap OR) \\ &= \bigcup_{\alpha < \beta} \alpha \quad \text{by induction hypothesis} \\ &= \beta \end{aligned}$$

Lemma

$L_\beta \cap OR = \beta$  for any ordinal  $\beta$

It is sufficient to show  $\beta \in D^+(L_\beta)$

Note:  $\beta = \{x \in L_\beta \mid x\}$ .

For  $\beta = \gamma + 1$

$$L_{\gamma+1} \cap OR = L_\gamma \cup D^+(L_\gamma) \cap OR$$

$$= \gamma \cup (D^+(L_\gamma) \cap OR)$$

$$\boxed{L_\gamma}$$

"  
 $\{\gamma\}$

$$L_{\gamma+1} \cap OR$$

$$\cong \bigvee_{\gamma+1} \cap OR = \gamma + 1$$

# Constructible sets

## Definition

For  $x \in L$ , we define the  $L$ -rank of  $x$ ,

$$\text{rank}_L(x) = \min \{ \alpha \in \text{OR} \mid x \in L_{\alpha+1} \}$$

# Constructible sets

## Fact

- $L_\alpha = \{x \in L \mid \text{rank}_L(x) < \alpha\}$
- $L_{\alpha+1} \setminus L_\alpha = \{x \in L \mid \text{rank}_L(x) = \alpha\}$
- $\text{rank}_L(L(\alpha)) = \text{rank}_L(\alpha) = \alpha$

# Constructible sets

## Fact

- Every finite subset of  $L_\alpha$  is in  $L_{\alpha+1}$
- $L_\alpha = V_\alpha$  for  $\alpha \leq \omega$
- $L_{\omega+1} \subsetneq V_{\omega+1}$

# Constructible sets

Theorem ((meta-theorem))

All axioms of ZF holds in  $L$

Lemma

Suppose that  $M$  is a transitive class and Comprehension Schema holds in  $M$ , and assume that for every subset  $x \subset M$ , there is a set  $y \in M$  such that  $x \subset y$ . Then All axioms of ZF holds in  $M$

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- Comprehension holds in  $L$
- for each  $x \in L$ , there is a  $y \in L$  such that  $x \subset y$

# Constructible sets

## Lemma

The function  $\alpha \rightarrow L_\alpha$  is absolute for any transitive class

$M \models \text{ZF} - \text{Pow}$

## Corollary

$(V = L)^L$

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## Corollary

$(V = L)^L$

- $L_\alpha$  is well-defined in  
transitive  $M \models \text{ZF} - \text{Pow}$
- $L_\alpha^M = L_\alpha$

# Next on Set Theory

- $V = L \models AC$
- $V = L \models GCH$