

Set Theory II

集合论 II

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Previously on Set Theory

- Δ_0 (module logical equivalence) formulas are absolute for transitive classes (no additional precondition)
- Δ_0 (module \wedge) formulas are absolute for transitive classes satisfying \wedge . It is also true for Δ_1 formulas
- Δ_0 or Δ_1 in parameters are absolute if the parameters are
- Recursive definitions preserve absoluteness
- **Functions** are absolute if it is still a function and absolute as a relation

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Constructible sets

Recall:

Informally, we define $\mathcal{D}(A, P)$ ($P \subset A$) to be the collection of all subsets X of A which are definable in (A, \in) from parameters from P , i.e. there exists a formula $\varphi(x, z_1, \dots, z_n)$ and $p_1, \dots, p_n \in P$ such that

$$X = \{a \in A \mid (A, \in) \models \varphi[a, p_1, \dots, p_n]\}$$

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Constructible sets

Formally, we define

$$\mathcal{D}(A, P) = \left\{ X \subset A \mid \exists n, k \in \omega \exists s \in P^k \left(n \text{ is a } \mathcal{L}_\epsilon \text{ formula with} \right. \right. \\ \left. \left. \begin{array}{l} \text{at most } k + 1 \text{ free variables} \\ \wedge X \text{ is defined by } n \text{ with parameters } s \text{ in } (A, \epsilon) \right) \right\}$$

$$\mathcal{D}^+(A) = \mathcal{D}(A, A)$$

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Constructible sets

Lemma (meta-theorem)

For each set theory formula $\varphi(x, z_1, \dots, z_n)$,

$$\text{ZF}^- - \text{Pow} \vdash \forall A \forall p_1, \dots, p_n \in A \left[\begin{array}{l} \{a \in A \mid \varphi^A(a, p_1, \dots, p_n)\} \in \mathcal{D}^+(A) \end{array} \right]$$

Constructible sets

Fact

Every finite subset of A is in $\mathcal{D}^+(A)$

Constructible sets

Definition

- $L_0 = \emptyset$
- $L_{\beta+1} = \mathcal{D}^+(L_\beta)$
- $L_\gamma = \bigcup_{\alpha < \gamma} L_\alpha$ for limit ordinal γ
- $L = \bigcup_{\alpha \in \text{OR}} L_\alpha$

$$V_0 = \emptyset$$

$$V_{\beta+1} = \mathcal{P}(V_\beta)$$

⋮

Constructible sets

Lemma

- $L_\alpha \subset V_\alpha$
- $L_\beta \subset L_{\beta+1}$
- L_α is transitive
- $\alpha \leq \beta \rightarrow L_\alpha \subset L_\beta$

Constructible sets

Induction on β

For limit case

$$\begin{aligned}L_\beta \cap \text{OR} &= \bigcup_{\alpha < \beta} L_\alpha \cap \text{OR} \\ &= \bigcup_{\alpha < \beta} (L_\alpha \cap \text{OR}) \\ &= \bigcup_{\alpha < \beta} \alpha \quad \text{by induction hypothesis}\end{aligned}$$

Lemma

$L_\beta \cap \text{OR} = \beta$ for any ordinal β

It is sufficient to show $\gamma \in D^+(L_\gamma)$

Note: $\gamma = \{x \in L_\gamma \mid x\}$

For $\beta = \gamma + 1$

$$L_{\gamma+1} \cap \text{OR} = L_\gamma \cup D^+(L_\gamma) \cap \text{OR}$$

$$= \gamma \cup (D^+(L_\gamma) \cap \text{OR})$$

$\{ \gamma \}$

$L_{\gamma+1} \cap \text{OR}$

$$\cong \bigvee_{\gamma+1} \cap \text{OR} = \gamma + 1$$

Constructible sets

Definition

For $x \in L$, we define the L -rank of x ,

$$\text{rank}_L(x) = \min \{ \alpha \in \text{OR} \mid x \in L_{\alpha+1} \}$$

Constructible sets

Fact

- $L_\alpha = \{x \in L \mid \text{rank}_L(x) < \alpha\}$
- $L_{\alpha+1} \setminus L_\alpha = \{x \in L \mid \text{rank}_L(x) = \alpha\}$
- $\text{rank}_L(L(\alpha)) = \text{rank}_L(\alpha) = \alpha$

Constructible sets

Fact

- Every finite subset of L_α is in $L_{\alpha+1}$
- $L_\alpha = V_\alpha$ for $\alpha \leq \omega$
- $L_{\omega+1} \subsetneq V_{\omega+1}$

Constructible sets

Theorem ((meta-theorem))

All axioms of ZF holds in L

Lemma

Suppose that M is a transitive class and Comprehension Schema holds in M , and assume that for every subset $x \subset M$, there is a set $y \in M$ such that $x \subset y$. Then All axioms of ZF holds in M

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- Comprehension holds in L
- for each $x \in L$, there is a $y \in L$ such that $x \subset y$

Constructible sets

Lemma

The function $\alpha \rightarrow L_\alpha$ is absolute for any transitive class

$M \models \text{ZF} - \text{Pow}$

Corollary

$(V = L)^L$

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Corollary

$(V = L)^L$

- L_α is well-defined in
transitive $M \models \text{ZF} - \text{Pow}$
- $L_\alpha^M = L_\alpha$

Next on Set Theory

- $V = L \models AC$
- $V = L \models GCH$