Set Theory II

集合论 II

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

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Previously on Set Theory

$M \subset N$

- φ is absolute for M, N if for any $a_1, \ldots, a_n \in M$,

  $\varphi^M(a_1, \ldots, a_n) \leftrightarrow \varphi^N(a_1, \ldots, a_n)$
- Δ_0 formula is absolute for transitive M
- A n-ary function is absolute if it is still a function in M and absolute as an $n+1$-relation

Previously on Set Theory

Basic set theory

$ZF - Inf - (Pow + Rep)$

Absolute in transitive models of BST:

- $\subset$ as 2-ary relation
- $\emptyset$ as 0-ary function
- $\bigcup$ as 1-ary function
- intersection $\quad x \cap y = z$

  int as 3-ary relation, $\cap$ as 2-ary function
- $\{x, y\}$ as 2-ary function, "being singleton" as 1-ary relation

Previously on Set Theory

If φ is Δ_0 in some $P_1, \ldots, P_n$ and $f_1, \ldots, f_m$, moreover, $P_1, \ldots, P_n$ and $f_1, \ldots, f_m$ are absolute for transitive M, then φ is also absolute

More on absoluteness

Lemma

Let φ, ψ be two formulas and $\forall \vec{x}[\varphi(\vec{x}) \leftrightarrow \psi(\vec{x})]$ holds in M and in N, then φ is absolute for M, N iff ψ is

Practically, if we have proved that both M and N satisfy Λ, and $\Lambda \vdash \forall \vec{x}[\varphi(\vec{x}) \leftrightarrow \psi(\vec{x})]$, then we can apply the lemma to M, N

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φ, ψ are equivalent modulo Λ

More on absoluteness

Lemma

The following notions are absolute for transitive models of BST

Foundation $\vdash$ x is well-ordered by $\in$ $\iff$ x is totally ordered by $\in$

$\forall y \in x \, \forall z \in y \, (z \in x)$

- x being a transitive set Δ_0 ✓

① be transitive ② wellordered by $\in$

- being an ordinal, successor ordinal, limit ordinal
- being a natural number x is ordinal $\wedge \forall y \leq x \, (y = \emptyset \vee y$ is succ. ordinal$)$
- $x \subset \omega$, $x = \omega$ $\quad x \subseteq \omega \wedge \emptyset \in x \wedge \forall z \in x \; Sz \in x$

$\forall y \in x \, (y$ is natural number$)$

More on absoluteness

Lemma

The following notions are absolute for transitive models of BST (Rep)

- the 2-ary ordered pair function $(x, y) = \{\{x\}, \{x, y\}\}$
- Cartesian product $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$
- being an ordered pair, being a relation
- the 1-ary functions $\operatorname{dom} x$ and $\operatorname{ran} x$
- being a function, being an injection, surjection, bijection

More on absoluteness

Lemma

The following notions are absolute for transitive models of BST

- the binary relation $\operatorname{apply}(f, x) = f(x)$
- R is a transitive, reflexive, total, symmetric, ... relation on A

More on absoluteness

Definition

Let the language $\mathcal{L}$ containing predicates $P_1, \ldots, P_n$ and function symbols $f_1, \ldots, f_m$, φ is a $\mathcal{L}$-formula. We say

- φ is Σ_1 (in $P_1, \ldots, P_n$ and $f_1, \ldots, f_m$) if φ is of the fom $\exists \vec{x} \psi$ where ψ is Δ_0 in (in $P_1, \ldots, P_n$ and $f_1, \ldots, f_m$)
- φ is Π_1 (in $P_1, \ldots, P_n$ and $f_1, \ldots, f_m$) if φ is of the fom $\forall \vec{x} \psi$ where ψ is Δ_0 in (in $P_1, \ldots, P_n$ and $f_1, \ldots, f_m$)

More on absoluteness

Lemma

Let M be transitive in N.

- Assume φ is Σ_1 in some notions which are absolute in M, N. Then φ is upward absolute for M, N, i.e.

$$\forall \vec{x}[\varphi^M(\vec{x}) \underline{\underline{\rightarrow}} \varphi^N(\vec{x})]$$

- If φ is Π_1 in these notions, then φ is downward absolute for M, N $\quad \forall \vec{x}\,[\varphi^M(\vec{x}) \leftarrow \varphi^N(\vec{x})]$

More on absoluteness

Let Λ be a set theory, we say φ is Δ_1 (in some notions) module Λ, if Λ proves that φ is equivalent to some Σ_1 (in some notions) formula and some Π_1 formula.

Lemma

Let M, N be models of Λ and M is transitive in N. Assume φ is Δ_1 module Λ where all the parameters are absolute for M, N. Then φ is absolute for M, N

More on absoluteness

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More on absoluteness

Lemma

The notions "R well-orders A" and "R is well-founded on A" are absolute for transitive models of ZF – Pow

"R well-orders A" is Δ_1 (in "order", "apply"...) modulo ZF − P

More on absoluteness

Fact

Let M be a transitive model of BST. Then

$= \{x \subseteq M \mid |x| < \omega\}$

- $[M]^{<\omega} \subset M$
- HF $\subset M$

  V_ω
- $M^{<\omega} \subset M$

  $\shortparallel$ $\{s \mid \exists n < \omega \; s : n \to M\}$

More on absoluteness

Lemma

"being finite", "being hereditarily finite" are absolute for transitive models of BST

n is natural number

$M \models BST$

x is finite $\leftrightarrow$ $\exists n, f \; (f : n \xrightarrow[\text{onto}]{1-1} x)$

$\to \exists n, f \in M \; (f : n \xrightarrow[\text{onto}]{1-1} x)$

More on absoluteness

Lemma

The following are absolute for transitive models of ZF – Pow

- the 0-ary founction HF
- the 0-ary founction ω
- the 1-ary function $[x]^{<\omega}$ and $x^{<\omega}$

Exe

More on absoluteness

Definition

An n-ary relation R is arithmetical if it is of the form $\left\{\vec{x} \in \mathrm{HF} \mid \mathrm{HF} \vDash \varphi(\vec{x})\right\}$ for some formula φ

Lemma

Every arithmetical relation is absolute for all transitive models of BST

e.g. Being a term, formula, sentence in the language of set theory

More on absoluteness

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More on absoluteness

Lemma

Let M be a transitive model of ZF – Pow. A, R, G are defined classes such that R is a well-founed, set-like relation on A, G is a 2-ary function. Assume A, R, G are all absolute for M, $(R$ is set-like on $A)^M$, and for each $a \in M$, $a\downarrow = \{x \mid xRa\} \subset M$. Let F be defined recursively by

$$\forall a \in A\big[F(a) = G(a, F\restriction(a\downarrow))\big] \quad \wedge \forall a \notin A\,\big(F(a) = \emptyset\big)$$

Then F is absolute for M

More on absoluteness

Corollary

The following notions are absolute for transitive models of ZF – Pow

- ordinal arithmetic funtion: $\alpha + \beta$, $\alpha \cdots \beta$, α^{β}
- Being a formula, sentence for possibly uncountable languages
- $\underline{\mathfrak{A} \models \varphi[\vec{c}]}$ $\quad$ $\mathrm{Sat}(\mathfrak{A}, \varphi, \vec{c})$ $\quad$ ZF
- $\mathcal{D}(A, P)$ = the set of all subsets of A that are definable over $(A, \in)$ with parameters in P

Next on Set Theory

- Constructible sets