

Set Theory II

集合论 II

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

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Previously on Set Theory

$$M \subset N$$

- φ is absolute for M, N if for any $a_1, \dots, a_n \in M$,
$$\varphi^M(a_1, \dots, a_n) \leftrightarrow \varphi^N(a_1, \dots, a_n)$$
- Δ_0 formula is absolute for transitive M
- A n -ary function is absolute if it is still a function in M and absolute as an $n + 1$ -relation

Previously on Set Theory

Absolute in transitive models of BST: *Basic set theory*
ZF - Int - (Power & Rep)

- \subset as 2-ary relation
- \emptyset as 0-ary function
- \cup as 1-ary function
- \cap as 2-ary function, \int as 3-ary relation, *int* as 3-ary relation, $x \cap y = \emptyset$
- $\{x, y\}$ as 2-ary function, “being singleton” as 1-ary relation

Previously on Set Theory

If φ is Δ_0 in some P_1, \dots, P_n and f_1, \dots, f_m , moreover, P_1, \dots, P_n and f_1, \dots, f_m are absolute for transitive M , then φ is also absolute

More on absoluteness

Lemma

Let φ, ψ be two formulas and $\forall \vec{x}[\varphi(\vec{x}) \leftrightarrow \psi(\vec{x})]$ holds in M and in N , then φ is absolute for M, N iff ψ is

Practically, if we have proved that both M and N satisfy Λ , and $\Lambda \vdash \forall \vec{x}[\varphi(\vec{x}) \leftrightarrow \psi(\vec{x})]$, then we can apply the lemma to M, N

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φ, ψ are equivalent modulo Λ

More on absoluteness

Lemma

The following notions are absolute for **transitive** models of

BST

- $\forall y \in x \forall z \in y (z \in x)$ Δ
 - being a transitive set Δ
 - ① be transitive ② wellfounded by \in
 - being an ordinal, successor ordinal, limit ordinal
 - being a natural number x is ordinal $\wedge \forall y \in x (y = \emptyset \vee y$ is succ. ordinal)
 - $x \subset \omega$, $x = \omega$ $\rightarrow x \subset \omega \wedge \emptyset \in x \wedge \forall z \in x \exists z \in x$
 - $\forall y \in x (y$ is natural number)
- Foundation \vdash x is well-ordered by \in
 $\Leftrightarrow x$ is totally ordered by \in

More on absoluteness

Lemma

The following notions are absolute for **transitive** models of **BST** (Rep)

- the 2-ary ordered pair function $(x, y) = \{ \{x, \{x, y\}\} \}$
- Cartesian product $X \times Y = \{ (x, y) \mid x \in X, y \in Y \}$
- being an ordered pair, being a relation
- the 1-ary functions $\text{dom } x$ and $\text{ran } x$
- being a function, being an injection, surjection, bijection

More on absoluteness

Lemma

The following notions are absolute for **transitive** models of **BST**

- the binary relation $\text{apply}(f, x) = f(x)$
- R is a transitive, reflexive, total, symmetric, ... relation on A

More on absoluteness

Definition

Let the language \mathcal{L} containing predicates P_1, \dots, P_n and function symbols f_1, \dots, f_m , φ is a \mathcal{L} -formula. We say

- φ is Σ_1 (in P_1, \dots, P_n and f_1, \dots, f_m) if φ is of the form $\exists \vec{x} \psi$ where ψ is Δ_0 (in P_1, \dots, P_n and f_1, \dots, f_m)
- φ is Π_1 (in P_1, \dots, P_n and f_1, \dots, f_m) if φ is of the form $\forall \vec{x} \psi$ where ψ is Δ_0 (in P_1, \dots, P_n and f_1, \dots, f_m)

More on absoluteness

Lemma

Let M be transitive in N .

- Assume φ is Σ_1 in some notions which are absolute in M, N . Then φ is **upward absolute** for M, N , i.e.

$$\forall \vec{x} [\varphi^M(\vec{x}) \rightarrow \varphi^N(\vec{x})]$$

- If φ is Π_1 in these notions, then φ is **downward absolute** for M, N

More on absoluteness

Let Λ be a set theory, we say φ is Δ_1 (in some notions) **module** Λ , if Λ proves that φ is equivalent to some Σ_1 (in some notions) formula and some Π_1 formula.

Lemma

Let M, N be models of Λ and M is transitive in N . Assume φ is Δ_1 module Λ where all the parameters are absolute for M, N .
Then φ is absolute for M, N

More on absoluteness

Let Λ be a set theory, we say φ is Δ_1 (in some notions) **module** Λ , if Λ proves that φ is equivalent to some Σ_1 (in some notions) formula and some Π_1 formula.

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Then φ is absolute for M, N

More on absoluteness

Lemma

The notions “ R well-orders A ” and “ R is well-founded on A ” are absolute for transitive models of ZF – Pow

More on absoluteness

Fact

Let M be a transitive model of BST. Then

- $[M]^{<\omega} \subset M$
- $\text{HF} \subset M$
- $M^{<\omega} \subset M$

More on absoluteness

Lemma

“being finite”, “being hereditarily finite” are absolute for transitive models of BST

More on absoluteness

Lemma

The following are absolute for transitive models of ZF – Pow

- the 0-ary function HF
- the 0-ary function ω
- the 1-ary function $[x]^{<\omega}$ and $x^{<\omega}$

More on absoluteness

Definition

An n -ary relation R is **arithmetical** if it is of the form

$\{\vec{x} \in \text{HF} \mid \text{HF} \models \varphi(\vec{x})\}$ for some formula φ

Lemma

Every arithmetical relation is absolute for all transitive models of BST

e.g. Being a term, formula, sentence in the language of set theory

More on absoluteness

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More on absoluteness

Lemma

Let M be a transitive model of $ZF - \text{Pow}$. A, R, G are defined classes such that R is a well-founded, set-like relation on A , G is a 2-ary function. Assume A, R, G are all absolute for M , (R is set-like on A) M , and for each $a \in M$, $a \downarrow = \{x \mid xRa\} \subset M$. Let F be defined recursively by

$$\forall a \in A \left[F(a) = G(a, F \upharpoonright (a \downarrow)) \right]$$

Then F is absolute for M

More on absoluteness

Corollary

The following notions are absolute for transitive models of
ZF – Pow

- ordinal arithmetic function: $\alpha + \beta$, $\alpha \cdot \beta$, α^β
- Being a formula, sentence for possibly uncountable languages
- $\mathfrak{A} \models \varphi[\vec{c}]$
- $\mathcal{D}(A, P)$ = the set of all subsets of A that are definable over (A, \in) with parameters in P

Next on Set Theory

- Constructible sets