Set Theory II

集合论 II

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Spring 2016

Previously on Set Theory

$ZF \vdash Con(ZF - inf)$

$ZF - inf \vdash Con(ZFC^-) \Rightarrow Con(ZFC)$

- $\Gamma \rhd \Lambda$, $\Gamma \leq \Lambda$
- Relativization

$M = \{x \mid \varphi_M(x)\}$

$\varphi \mapsto \varphi^M$

$M \models \varphi$

Previously on Set Theory

To show $\mathrm{Con}(\mathrm{ZFC}^-) \rightarrow \mathrm{Con}(\mathrm{ZFC})$, we prove finitistically that

$$\mathrm{ZFC}^- \vdash \sigma^{\mathrm{WF}}$$

for each σ in ZFC

Previously on Set Theory

Lemma

For any class M,

- If M is transitive, then Ext^M
- If $M \subset \mathrm{WF}$, then $\mathrm{Foundation}^M$ EXE:
- If $\forall z \in M \forall y \subset z (y \in M)$, then $\mathrm{Separation}^M$
- If $\forall x, y \in M(\{x, y\} \in M)$, then Pair^M EXE:
- If $\forall X \in M(\bigcup X \in M)$, then Union^M EXE:
- If M is transitive and for all function f, $\operatorname{dom} f \in M$ and $\operatorname{ran} f \subset M$ imply $\operatorname{ran} f \in M$, then Rep^M

Class model

Lemma

Let M be a transitive class. Then

$$\forall x \in M\Big((P(x) \cap M) \in M\Big) \to \mathrm{Pow}^M$$

$$\left[\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x)\right]^M$$

$$= \forall x \in M \, \exists y \in M \, \forall z \in M \, (z \in y \leftrightarrow z \subseteq x)$$

y: $P(x) \cap M$

$z \subseteq x$: $\forall w \in M (w \in z \to w \in x)$

It is easy if we can show

$$\forall x, z \in M\Big((z \subset x)^M \leftrightarrow (z \subset x)\Big)$$

$(\to)$ $z \in M \cap P(x) \to z \subseteq x$

$(\leftarrow)$ $z \in M \wedge z \subseteq x \to z \in P(x) \cap M$

Class model

Lemma

Let M be a transitive class. Then

$$\forall x \in M\big((P(x) \cap M) \in M\big) \to \mathrm{Pow}^M$$

It is easy if we can show

$\forall w \in M (w \in z \to w \in x)$

$$\forall x, z \in M\big((z \subset x)^M \leftrightarrow (z \subset x)\big)$$

M is transitive. so $z \in M \Rightarrow z \subseteq M$, so if $w \in z$, then, $w \in M$. $\forall w (w \in z \to w \in x)$

Absoluteness

Definition (In metalanguage)

ZF ⊢ ∀x (φ_M(x) → φ_N(x))

Let $M \subset N$ be classes, and $\varphi(x_1, \ldots, x_n)$ is a formula, we say φ is absolute for M, N if for any $a_1, \ldots, a_n \in M$

$$\varphi^M(a_1, \ldots, a_n) \leftrightarrow \varphi^N(a_1, \ldots, a_n)$$

Is provable in ZF/ZFC

Absoluteness

Definition (In metalanguage)

Let $M \subset N$ be classes, and $\varphi(x_1, \ldots, x_n)$ is a formula, we say φ is absolute for M if for any $a_1, \ldots, a_n \in M$

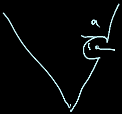
$$\varphi^M(a_1, \ldots, a_n) \leftrightarrow \varphi \ (a_1, \ldots, a_n)$$

Is provable in ZF/ZFC

Absoluteness

Example

- For any $M \subset N$, $x = y, x \in y$ is absolute for M, N
- If M is transitive in N, then $x \subset y$ is absolute for M, N
- If M is not transitive, then $x \subset y$ may be not absolute for M



Definition (In metalanguage)

Δ_0 formula is recursively defined as follow

- Atomic formula, say $x = y, x \in y$ are Δ_0
- If φ and ψ are Δ_0, then $\neg\varphi$, $\varphi \to \psi$, etc. are Δ_0
- If φ is Δ_0 and $x \neq y$, then $\forall x \in y \varphi$ and $\exists x \in y \varphi$ are Δ_0

Intuitively, Δ_0 formulas represent local properties / relations

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Absoluteness

$\forall z(\exists w(x\in y \wedge z\in w) \to z\in x)$

$\forall z \forall w (w\in y \wedge z\in w \to z\in x)$

$\forall w\in y\, \forall z\in w\; z\in x$

Example

$\forall z\in x\,(z\in y)$

$\forall z\,(z\in x \to z\in y)$

- $x \subset y$ ($x \in P(y)$) is Δ_0
- $x = P(y)$ is not Δ_0 $\forall z\,(z\in x \leftrightarrow z\subseteq y)$

  $\forall z\,(z\in x \to z\subseteq y) \wedge \forall z\,(z\subseteq y \to z\in x)$
- $x = \emptyset$ $\forall y\in x\,(y\neq y)$
- $x = \bigcup y$ is logically equivalent to a Δ_0 formula

  $\forall z\,(z\in x \leftrightarrow \exists w\,(w\in y \wedge z\in w))$ $\forall z\in x\, \exists w\in y\; z\in w \wedge$

Absoluteness

Lemma (In metalanguage)

Given classes $M \subset N$, such that M is transitive in N, and φ (logically equivalent to) a Δ_0 formula. Then φ is absolute for M, N

Absoluteness

Example

- $y = Sx$ $\quad \forall z\,(z \in y \leftrightarrow z = x \vee z \in x)$

  $\forall z \in y\,(z = x \vee z \in x) \wedge \forall z \in x\,(z \in y) \wedge x \in y$

- $z = x \cap y$ $\quad \mathrm{Int}(x, y, z)$ $\quad \forall w \in z\,(w \in x \wedge w \in y) \wedge \forall w \in x\,(w \in y \to w \in z)$

- x is singleton

  $\exists y \in x\, \forall z \in x\,(y = z)$

Absoluteness

Convention

Let P be an n-ary predicate defined by some formula φ_P, i.e.$\forall x_1, \ldots, x_n [P(x_1, \ldots, x_n) \leftrightarrow \varphi_P(x_1, \ldots, x_n)]$. Then the relativization $P^M(x_1, \ldots, x_n)$ is $\varphi_P^M(x_1, \ldots, x_n)$. We say P is absolute for M, N if φ_M is

Absoluteness

Example

If M is transitive, then $\subset$ is absolute for M

Convention

Let f be an n-ary function symbol introduced by the definition $\forall x_1, \ldots, x_n [\varphi_f(x_1, \ldots, x_n, f(x_1, \ldots, x_n))]$, and suppose we have proved $\left[\forall x_1, \ldots, x_n \exists! y \varphi_f(x_1, \ldots, x_n, y)\right]^M$. Then for $x_1, \ldots, x_n \in M$, $f^M(x_1, \ldots, x_n)$ denotes the unique $y \in M$ such that $\varphi_f^M(x_1, \ldots, x_n, y)$. We say f is absolute for M, N if $f^M(\vec{x}) = f^N(\vec{x})$ for all $x_1, \ldots, x_n \in M$.

Absoluteness

Example

Both the function $v \cap w = y$ and the relation $\mathsf{int}(v, w, y)$ is introduced by the formula $\forall x(x \in y \leftrightarrow x \in v \wedge x \in w)$. Since it is Δ_0, int is absolute for any transitive class M. But $\cap$ is absolute iff $\cap^M$ is defined, i.e. for each $v, w \in M$, there is a unique y such that $\mathrm{int}^M(x, w, y)$, and int is absolute for M

Absoluteness

Example

- Let $M = \mathrm{OR} \cup \left\{(\alpha, \beta) \;\middle|\; \alpha < \beta \in \mathrm{OR}\right\}$, then M is transitive and $\cap^M$ is not defined
- Let $M = \left\{2, \{1, 2\}\right\}$, then $\cap^M$ is defined, but $\cap$ is not absolute for M

Absoluteness

Lemma

If M is a model for the axioms of extensionality, separation, pairing and union, then $\emptyset^M$, S^M, $\cap^M$ are defined. Furthermore, if M is also transitive, then these are absolute for M

Absoluteness

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Absoluteness

Lemma ($ZF^- - Pow$)

Let M be a transitive class satisfying extensionality, separation, pairing and union. Then

- Inf^M if $\omega \in M$
- AC^M if every disjoint family of non-empty sets in M has a choice set in M

Class model

Fact

$\mathrm{Con}(\mathrm{ZFC}^{-}) \rightarrow \mathrm{Con}(\mathrm{ZFC})$

Definition

Let the language $\mathcal{L}$ containing predicates $P_1, \ldots, P_n$ and function symbols $f_1, \ldots, f_m$, φ is a $\mathcal{L}$-formula. We say φ is Δ_0 (in $P_1, \ldots, P_n$ and $f_1, \ldots, f_m$) if

- φ is atomic
- φ is a boolean composition of Δ_0 formulas
- ψ is Δ_0, τ is a term does not contain y and $\varphi = \exists y \in \tau\psi$ or $\forall y \in \tau\psi$

Absoluteness

Lemma

Predicates $P_1, \ldots, P_n$ and function symbols $f_1, \ldots, f_m$ are introduced by definitions in the language of set theory, φ is Δ_0 in $P_1, \ldots, P_n$ and $f_1, \ldots, f_m$, M is transitive in N. Now, if $P_1, \ldots, P_n$ and $f_1, \ldots, f_m$ are absolute for M, N, then φ is also absolute in M, N

EXE:

Next on Set Theory

- More on absoluteness
- Constructible sets