

Set Theory II

集合论 II

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Previously on Set Theory

$$\exists \bar{F} \vdash \text{Con}(\exists \bar{F} - \text{inf})$$

$$\exists \bar{F} - \text{inf} \vdash \text{Con}(\exists \bar{F} \text{C}^-) \rightarrow \text{Con}(\exists \bar{F} \text{C})$$

■ $\Gamma \triangleright \Lambda$, $\Gamma \leq \Lambda$

■ Relativization

$$\varphi \mapsto \varphi^M$$

$$M = \{x \mid \varphi_m(x)\}$$

$$M \models \varphi$$

Previously on Set Theory

ZFC

To show $\text{Con}(\text{ZFC}^-) \rightarrow \text{Con}(\text{ZFC})$, we prove finitistically that

for each σ in ZFC

ZFC $\vdash \sigma$

ZFC $\vdash \neg \sigma$

$\text{ZFC}^- \vdash \sigma^{\text{WF}}$

$(x \neq x)^{\text{WF}}$
 $\hline x \neq x$

Previously on Set Theory

Lemma

For any class M ,

- If M is transitive, then Ext^M
- If $M \subset \text{WF}$, then Foundation^M EXE:
- If $\forall z \in M \forall y \subset z (y \in M)$, then Separation^M
- If $\forall x, y \in M (\{x, y\} \in M)$, then Pair^M EXE:
- If $\forall X \in M (\cup X \in M)$, then Union^M EXE:
- If M is transitive and for all function f , $\text{dom } f \in M$ and $\text{ran } f \subset M$ imply $\text{ran } f \in M$, then Rep^M

Class model

Lemma

Let M be a transitive class. Then

$$\forall x \in M ((P(x) \cap M) \in M) \rightarrow \underline{\text{Pow}}^M$$

$$\begin{aligned} & \left[\forall x \exists y \forall z (z \in y \leftrightarrow z \in x) \right]^M \\ &= \forall x \in M \exists y \in M \forall z \in M (z \in y \leftrightarrow z \in x) \\ & \quad \underbrace{\quad \quad \quad}_{P(x) \cap M} \quad \underbrace{\quad \quad \quad}_{\forall w \in M (w \in y \rightarrow w \in x)} \end{aligned}$$

It is easy if we can show

$$\forall x, z \in M ((z \subset x)^M \leftrightarrow (z \subset x)) \quad \begin{array}{l} (\rightarrow) \quad z \in M \cap P(x) \rightarrow z \in x \\ (\leftarrow) \quad z \in M \cap z \subset x \rightarrow z \in \underline{P(x) \cap M} \end{array}$$

Class model

Lemma

Let M be a transitive class. Then

$$\forall x \in M \left((P(x) \cap M) \in M \right) \rightarrow \text{Pow}^M$$

It is easy if we can show

$$\forall w \in M (w \subseteq z \rightarrow w \in x)$$

$$\forall x, z \in M \left((z \subseteq x)^M \leftrightarrow (z \subseteq x) \right)$$

M is transitive, so $z \in M \Rightarrow z \subseteq M$, so if $w \in z$, then, $w \in M$. $\forall w (w \subseteq z \rightarrow w \in x)$

Absoluteness

/ Definition (In metalanguage)

$ZF \vdash \forall x (\varphi_{ind}(x) \rightarrow \varphi_{tr}(x))$

Let $M \subset N$ be classes, and $\varphi(x_1, \dots, x_n)$ is a formula, we say φ is **absolute for M, N** if for any $a_1, \dots, a_n \in M$

$$\varphi^M(a_1, \dots, a_n) \leftrightarrow \varphi^N(a_1, \dots, a_n)$$

Is provable in ZF/ZFC

Absoluteness

Definition (In metalanguage)

Let $M \subset N$ be classes, and $\varphi(x_1, \dots, x_n)$ is a formula, we say φ is **absolute for M** if for any $a_1, \dots, a_n \in M$

$$\underline{\varphi^M(a_1, \dots, a_n)} \leftrightarrow \varphi(a_1, \dots, a_n)$$

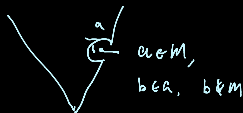
Is **provable in ZF/ZFC**

Absoluteness

Example

- For any $M \subset N$, $x = y, x \in y$ is absolute for M, N
 $\forall x \in M (x \cap N \subseteq x \cap M)$
- If M is transitive in N , then $x \subset y$ is absolute for M, N
- If M is not transitive, then $x \subset y$ may be **not** absolute for

M



$$a \setminus \{b\} \in M \quad \forall x \in M (x \in a \rightarrow x \in a \setminus \{b\})$$
$$\underline{M \not\models a \subseteq a \setminus \{b\}}$$

Absoluteness

Definition (In metalanguage)

Δ_0 formula is recursively defined as follow

- Atomic formula, say $x = y, x \in y$ are Δ_0
- If φ and ψ are Δ_0 , then $\neg\varphi, \varphi \wedge \psi, \varphi \rightarrow \psi$, etc. are Δ_0
- If φ is Δ_0 and $x \neq y$, then $\forall x \in y \varphi$ and $\exists x \in y \varphi$ are Δ_0

Intuitively, Δ_0 formulas represent **local** properties / relations

Absoluteness

Definition (In metalanguage)

Δ_0 formula is recursively defined as follow

- Atomic formula, say $x = y, x \in y$ are Δ_0
- If φ and ψ are Δ_0 , then $\neg\varphi, \varphi \rightarrow \psi$, etc. are Δ_0
- If φ is Δ_0 and $\underline{x \neq y}$, then $\forall x \in y \varphi$ and $\exists x \in y \varphi$ are Δ_0

Intuitively, Δ_0 formulas represent **local** properties / relations

Absoluteness

Example

$$\forall z \in V (z \in y)$$

$$\forall z (z \in x \rightarrow z \in y)$$

$$\forall z (\exists w (x \in y \wedge z \in w) \rightarrow z \in x)$$

$$\forall z \forall w (w \in y \wedge z \in w \rightarrow z \in x)$$

$$\forall w \in y \forall z \in w \quad z \in x$$

■ $x \subset y$ ($x \in P(y)$) is Δ_0

■ $x = P(y)$ is **not** Δ_0 $\forall z (z \in x \leftrightarrow z \in y)$

■ $x = \emptyset$ $\forall y \in x (y \neq y)$ $\forall z (z \in x \leftrightarrow z \in y) \wedge \forall z (z \in y \rightarrow z \in x)$

■ $x = \bigcup y$ is **logically equivalent to** a Δ_0 formula

$$\forall z (z \in x \leftrightarrow \exists w (w \in y \wedge z \in w)) \quad \forall z \in x \exists w \in y \quad z \in w \wedge$$

Absoluteness

Lemma (In metalanguage)

Given classes $M \subset N$, such that M is transitive in N , and φ (logically equivalent to) a Δ_0 formula. Then φ is absolute for M, N

Absoluteness

Example

- $y = Sx$

- $z = x \cap y$

- x is singleton

$$\forall z (z \in y \leftrightarrow z = x \vee z \in x)$$

$$\forall z \in y (z = x \vee z \in x) \wedge \forall z \in x (z \in y) \wedge x \in y$$

Absoluteness

Convention

Let P be an n -ary **predicate** defined by some formula φ_P ,
i.e. $\forall x_1, \dots, x_n [P(x_1, \dots, x_n) \leftrightarrow \varphi_P(x_1, \dots, x_n)]$. Then the
relativization $P^M(x_1, \dots, x_n)$ is $\varphi_P^M(x_1, \dots, x_n)$. We say P is
absolute for M, N if φ_M is

Absoluteness

Example

If M is transitive, then \subset is absolute for M

Absoluteness

Convention

Let f be an n -ary **function** symbol introduced by the definition $\forall x_1, \dots, x_n [\varphi_f(x_1, \dots, x_n, f(x_1, \dots, x_n))]$, and suppose we have proved $\left[\forall x_1, \dots, x_n \exists! y \varphi_f(x_1, \dots, x_n, y) \right]^M$. Then for $x_1, \dots, x_n \in M$, $f^M(x_1, \dots, x_n)$ denotes the unique $y \in M$ such that $\varphi_f^M(x_1, \dots, x_n, y)$. We say f is **absolute for M, N** if $f^M(\vec{x}) = f^N(\vec{x})$ for all $x_1, \dots, x_n \in M$.

Absoluteness

Example

Both the function $v \cap w = y$ and the relation $\text{int}(v, w, y)$ is introduced by the formula $\forall x(x \in y \leftrightarrow x \in v \wedge x \in w)$. Since it is Δ_0 , int is absolute for any transitive class M . But \cap is absolute iff \cap^M is defined, i.e. for each $v, w \in M$, there is a unique y such that $\text{int}^M(x, w, y)$, and int is absolute for M

Absoluteness

Example

- Let $M = \text{OR} \cup \{(\alpha, \beta) \mid \alpha < \beta \in \text{OR}\}$, then M is transitive and \cap^M is not defined
- Let $M = \{2, \{1, 2\}\}$, then \cap^M is defined, but \cap is not absolute for M

Absoluteness

Lemma

If M is a model for the axioms of extensionality, separation, pairing and union, then \emptyset^M , S^M , \cap^M are defined. Furthermore, if M is also transitive, then these are absolute for M

Absoluteness

Lemma

If M is a model for the axioms of extensionality, **separation**, pairing and union, then \emptyset^M , S^M , \cap^M are defined. Furthermore, if M is also transitive, then these are absolute for M

Absoluteness

Lemma ($ZF^- - Pow$)

Let M be a transitive class satisfying extensionality, separation, pairing and union. Then

- Inf^M if $\omega \in M$
- AC^M if every disjoint family of non-empty sets in M has a choice set in M

Class model

Fact

$\text{Con}(\text{ZFC}^-) \rightarrow \text{Con}(\text{ZFC})$

Absoluteness

Definition

Let the language \mathcal{L} containing predicates P_1, \dots, P_n and function symbols f_1, \dots, f_m , φ is a \mathcal{L} -formula. We say φ is Δ_0 (in P_1, \dots, P_n and f_1, \dots, f_m) if

- φ is atomic
- φ is a boolean composition of Δ_0 formulas
- ψ is Δ_0 , τ is a term **does not contain y** and $\varphi = \exists y \in \tau \psi$
or $\forall y \in \tau \psi$

Absoluteness

Lemma

Predicates P_1, \dots, P_n and function symbols f_1, \dots, f_m are introduced by definitions in the language of set theory, φ is Δ_0 in P_1, \dots, P_n and f_1, \dots, f_m , M is transitive in N . Now, if P_1, \dots, P_n and f_1, \dots, f_m are absolute for M, N , then φ is also absolute in M, N

EXE:

Next on Set Theory

- More on absoluteness
- Constructible sets