Set Theory II

集合论 II

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Spring 2016

Previously on Set Theory

- $\mathsf{HF} = V_\omega$ (axiom of foundation)
- $\mathsf{HF} \models \mathsf{ZF} - \mathsf{Inf} + \neg\mathsf{Inf}$
- $\mathsf{ZF} \vdash \mathrm{Con}(\mathsf{ZF} - \mathsf{Inf} + \neg\mathsf{Inf})$

Inf

EXE: ZF+ there exists an inaccessible cardinal $\vdash$ Con(ZF)

Definition

Let Γ, Λ be two sets of sentences in the language of set theory extending $\mathrm{ZF} - \mathrm{Inf} - \mathrm{Pow}$. We say Γ is proof-theoretically strictly stronger than Λ (written $\Gamma \rhd \Lambda$) iff $\Gamma \vdash \mathrm{Con}(\Lambda)$

Example

- $\mathrm{ZF} \rhd (\mathrm{ZF} - \mathrm{Inf} + \neg\mathrm{Inf})$
- ($\mathrm{ZF}+$ there exists an inaccessible cardinal) $\rhd \mathrm{ZF}$
- $\mathrm{ZFC} \rhd (\mathrm{ZFC} - \mathrm{Pow})$

  EXE: Define hereditarily countable set HC within ZFC, and show that $\mathrm{HC} \vDash (\mathrm{ZFC} - \mathrm{Pow})$

Corollary

$\rhd$ is not reflexive

Definition

We define $\Gamma \leq \Lambda$ iff we have a finitistic proof that $Con(\Lambda) \rightarrow Con(\Gamma)$. We say Λ and Γ are **proof-theoretically equivalent** (written $\Gamma \sim \Lambda$) iff $\Gamma \leq \Lambda$ and $\Lambda \leq \Gamma$

Clearly, $\leq$ is transitive and reflexive

Corollary

$\triangleright$ is not reflexive

Definition

PA, ZF - Inf

We define $\Gamma \leq \Lambda$ iff we have a finitistic proof that $\mathrm{Con}(\Lambda) \to \mathrm{Con}(\Gamma)$. We say Λ and Γ are proof-theoretically equivalent (written $\Gamma \sim \Lambda$) iff $\Gamma \leq \Lambda$ and $\Lambda \leq \Gamma$

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Example (To be proved)

- PA $\sim$ (ZF – Inf)
- $\underline{\mathsf{ZF}^-} \sim \mathsf{ZF} \sim \mathsf{ZFC} \sim (\mathsf{ZFC} + \underline{\mathsf{CH}}) \sim (\mathsf{ZFC} + \neg\mathsf{CH})$

ZF – Foundation

$2^{\aleph_0} = \aleph_1$

Lemma

- $\Gamma \overset{\dashv}{\subset} \Lambda \to \Gamma \leq \Lambda$

  $\Rightarrow T \leq \Delta$

- $\Gamma \subset \Lambda \wedge \Lambda \leq \Gamma \to \Gamma \sim \Lambda$

  $\Delta \vdash Con(T)$ $\qquad Con(\Delta) \to Con(T)$

- $\Gamma \lhd \Lambda \to \Gamma \underline{\leq \Lambda}$
- $\Gamma \leq \Lambda \wedge \Lambda \lhd \Theta \to \overset{T}{\cancel{\Lambda}} \lhd \Theta$
- $\lhd$ is transitive

Proof Assume $\neg Con(T)$

i.e. $\underline{\exists \delta \; T \vdash \delta \wedge \neg \delta}$ $\quad \Sigma_1^0$

Since $\Delta \supseteq ZF\text{-inf}$ (PA)

So $\Delta \vdash \neg Con(T)$

And $\Delta \vdash Con(T)$, so

$\neg Con(\Delta)$

$\Delta \vdash Con(T)$

Gödel's completeness theorem says: if $\Gamma \lhd \Lambda$ is provable, then you can always prove it by finding a set model M of Γ working in a model of Λ

$\Delta \vdash \exists m \ m \models T$

While Gödel's incompleteness theorem says: If you want to prove $\Gamma \leq \Lambda$ with $\Lambda \subset \Gamma$, working in a universe of Λ, you should not found a set model of Γ

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Class model

Recall: In metalanguage, A class A is always of the form $\{x \mid \varphi_A(x)\}$ for some formula φ_A in the language of set theory

Example (proper class)

- $V = \{x \mid x = x\}$
- $\in = \{(x, y) \mid x \in y\}$
- $\mathrm{OR} = \{\alpha \mid \alpha \text{ is transitive and well ordered by } \in\}$
- EXE: $\mathrm{WF} = \bigcup_{\alpha \in \mathrm{OR}} V_\alpha = ?$

Class model

To prove: $\mathrm{Con}(\mathrm{ZFC}^-) \to \mathrm{Con}(\mathrm{ZFC})$

Intuitively, we assume $V \vDash \mathrm{ZFC}^-$, and show $\mathrm{WF} \vDash \mathrm{ZFC}$

But we cannot even say $V \vDash \mathrm{ZFC}^-$ or $\mathrm{WF} \vDash \mathrm{ZFC}$ in set theory

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To prove: $\mathrm{Con}(\mathrm{ZFC}^-) \to \mathrm{Con}(\mathrm{ZFC})$

Intuitively, we assume $V \models \mathrm{ZFC}^-$, and show $\mathrm{WF} \models \mathrm{ZFC}$

But we cannot even say $V \models \mathrm{ZFC}^-$ or $\mathrm{WF} \models \mathrm{ZFC}$ in set theory

Class model

Definition (in meta theory)

Let $M = \{x \mid \varphi_M(x)\}$ be a class and $E = \{(x, y) \mid \varphi_E(x, y) \wedge \varphi_M(x) \wedge \varphi_M(y)\}$ be a binary relation on M. We define recursively on the formulas of set theory

Class model

Definition (in meta theory)

Let M be a class and E be a binary relation on M. We define recursively on the formulas of set theory

Class model

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Let M be a class and E be a binary relation on M. We define recursively on the formulas of set theory

$$(x = y)^{M,E} =_{\text{df}} (x = y)$$

$$(x \in y)^{M,E} =_{\text{df}} \varphi_E(x, y)$$

$(M, E) \models x \in y$ iff $(x, y) \in E$

$$(\neg\alpha)^{M,E} =_{\text{df}} \neg\alpha^{M,E}$$

$$(\alpha \to \beta)^{M,E} =_{\text{df}} \alpha^{M,E} \to \beta^{M,E}$$

$$(\forall x\alpha)^{M,E} =_{\text{df}} \forall x(\varphi_M(x) \to \alpha^{M,E})$$

Class model

Convention

- In case $E = \in \cap M$, we write φ^{M} instead of $\varphi^{M, \in \cap M}$
- We say E is well-founded on M if for each non-empty subset A of M there is an E-minimal element $a \in A$
- We say E is set-like on M if for each $a \in M$, $\left\{ b \in M \,\middle|\, bEa \right\}$ is a set

$\forall x (x \in \alpha \to \varphi_{m}(x))$

- We say M is transitive if for each $a \in M$, $a \subset M$

Class model

Theorem (Mostowski collapse)

Let E be well-founded and set-like on M, then we can recursively define a transitive class N such that $(M, E) \simeq (N, \in)$

In this case, we say N is the Mostowski collapse of (M, E)

Class model

Recall:

Lemma (Soundness)

$\sigma \vdash \delta$ implies $\sigma \vDash \delta$, i.e. for each structure $\mathfrak{A}$, if $\mathfrak{A} \vDash \sigma$, then $\mathfrak{A} \vDash \delta$

By a similar argument:

Lemma (PA or ZF − Inf)

If $\sigma \vdash \delta$, then for any formula φ_M, φ_E, $\sigma^{M,E} \vdash \delta^{M,E}$

Class model

To show Con(ZFC$^-$) $\rightarrow$ Con(ZFC), we prove finitistically that

$$\mathrm{ZFC}^- \vdash \sigma^{\mathrm{WF}}$$

for each σ in ZFC

$\mathrm{ZFC}^- \vdash \mathrm{ZFC}^{\mathrm{WF}}$

And then $\underbrace{\mathrm{ZFC} \vdash x \neq x}_{\neg \mathrm{Con}(\mathrm{ZFC})}$ implies $\underbrace{\mathrm{ZFC}^- \vdash (x \neq x)^{\mathrm{WF}}}_{\neg \mathrm{Con}(\mathrm{ZFC}^-)}$ $= \underline{x \neq x}$

Class model

To show Con(ZFC–) $\to$ Con(ZFC), we prove finitistically that

$$\mathrm{ZFC}^{-} \vdash \sigma^{\mathrm{WF}}$$

for each σ in ZFC

And then $\mathrm{ZFC} \vdash x \neq x$ implies $\mathrm{ZFC}^{-} \vdash (x \neq x)^{\mathrm{WF}}$

Lemma

For any class M,

- If M is transitive, then Ext^M
- If $M \subset \mathrm{WF}$, then $\mathrm{Foundation}^M$
- If $\forall z \in M \forall y \subset z (y \in M)$, then $\mathrm{Separation}^M$
- If $\forall x, y \in M (\{x, y\} \in M)$, then Pair^M
- If $\forall X \in M (\bigcup X \in M)$, then Union^M
- If M is transitive and for all function f, $\operatorname{dom} f \in M$ and $\operatorname{ran} f \subset M$ imply $\operatorname{ran} f \in M$, then Rep^M

Lemma

For any class M,

- If M is transitive, then Ext^M
- If $M \subset \text{WF}$, then Foundation^M EXE:
- If $\forall z \in M \forall y \subset z (y \in M)$, then Separation^M
- If $\forall x, y \in M(\{x, y\} \in M)$, then Pair^M EXE:
- If $\forall X \in M(\bigcup X \in M)$, then Union^M EXE:
- If M is transitive and for all function f, $\text{dom} f \in M$ and $\text{ran} f \subset M$ imply $\text{ran} f \in M$, then Rep^M

Class model

Lemma

Let M be a transitive class. Then

$$\forall x \in M\Big((P(x) \cap M) \in M\Big) \to \mathrm{Pow}^M$$

Next on Set Theory

- Absoluteness

I.16

II.4