

Set Theory II

集合论 II

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Previously on Set Theory

- $\text{HF} = V_\omega$ (axiom of foundation)
- $\text{HF} \models \text{ZF} - \text{Inf} + \neg\text{Inf}$
- $\text{ZF} \vdash \text{Con}(\text{ZF} - \text{Inf} + \neg\text{Inf})$

\downarrow
 Inf

EXE: $ZF+$ there exists an inaccessible cardinal $\vdash \text{Con}(ZF)$

Definition

Let Γ, Λ be two sets of sentences in the language of set theory extending $ZF - \text{Inf} - \text{Pow}$. We say Γ is **proof-theoretically strictly stronger** than Λ (written $\Gamma \triangleright \Lambda$) iff $\Gamma \vdash \text{Con}(\Lambda)$

Example

- $ZF \triangleright (ZF - \text{Inf} + \neg \text{Inf})$
- $(ZF + \text{there exists an inaccessible cardinal}) \triangleright ZF$
- $ZFC \triangleright (ZFC - \text{Pow})$

EXE: Define **hereditarily countable set** HC within ZFC,
and show that $HC \models (ZFC - \text{Pow})$

Corollary

▷ is not reflexive

Definition

We define $\Gamma \leq \Lambda$ iff we have a **finitistic proof** that

$\text{Con}(\Lambda) \rightarrow \text{Con}(\Gamma)$. We say Λ and Γ are proof-theoretically equivalent (written $\Gamma \sim \Lambda$) iff $\Gamma \leq \Lambda$ and $\Lambda \leq \Gamma$

Clearly, \leq is transitive and reflexive

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Definition

PA, ZF - Int

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Example (To be proved)

■ $PA \sim (ZF - Inf)$

■ $ZF^- \sim ZF \sim ZFC \sim (ZFC + CH) \sim (ZFC + \neg CH)$

$\overline{ZF^-}$ Foundation

\overline{ZFC} = \overline{CH}

Lemma

$$\blacksquare \Gamma \subset \Lambda \xrightarrow{=} \Gamma \leq \Lambda$$

$$\Rightarrow \tau \leq \alpha$$

$$\blacksquare \Gamma \subset \Lambda \wedge \Lambda \leq \Gamma \rightarrow \Gamma \sim \Lambda$$

$$\Delta \vdash \text{Con}(\Gamma)$$

$$\text{Con}(\alpha) \rightarrow \text{Con}(\Gamma)$$

$$\blacksquare \Gamma \triangleleft \Lambda \rightarrow \underline{\Gamma \leq \Lambda}$$

$$\blacksquare \Gamma \leq \Lambda \wedge \Lambda \triangleleft \Theta \rightarrow \Gamma \triangleleft \Theta$$

■ \triangleleft is transitive

Proof Assume $\neg \text{Con}(\Gamma)$

is. $\exists \alpha \tau \vdash \alpha \wedge \neg \alpha \quad \Sigma_i$

Since $\Delta \geq \exists \bar{\tau} \text{-int}$ (PA)

so $\Delta \vdash \neg \text{Con}(\Gamma)$

And $\Delta \vdash \text{Con}(\Gamma)$, so

$\neg \text{Con}(\Delta)$

Gödel's completeness theorem says: if $\Gamma \triangleleft \Lambda$ is provable, then you can always prove it by finding a set model M of Γ working in a model of Λ

While Gödel's incompleteness theorem says: If you want to prove $\Gamma \leq \Lambda$ with $\Lambda \subset \Gamma$, working in a universe of Λ , you should **not** find a set model of Γ

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Class model

Recall: In metalanguage, A **class** A is always of the form $\{x \mid \varphi_A(x)\}$ for some formula φ_A in the language of set theory

Example (proper class)

- $V = \{x \mid x = x\}$
- $\in = \{(x, y) \mid x \in y\}$
- $\text{OR} = \{\alpha \mid \alpha \text{ is transitive and well ordered by } \in\}$
- EXE: $\text{WF} = \bigcup_{\alpha \in \text{OR}} V_\alpha = ?$

Class model

To prove: $\text{Con}(\text{ZFC}^-) \rightarrow \text{Con}(\text{ZFC})$

Intuitively, we assume $V \models \text{ZFC}^-$, and show $WF \models \text{ZFC}$

But we cannot even say $V \models \text{ZFC}^-$ or $WF \models \text{ZFC}$ in set theory

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Class model

Definition (in meta theory)

Let $M = \{x \mid \varphi_M(x)\}$ be a class and

$E = \{(x, y) \mid \varphi_E(x, y) \wedge \varphi_M(x) \wedge \varphi_M(y)\}$ be a binary relation on

M . We define recursively on the formulas of set theory

Class model

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Let M be a class and E be a binary relation on M . We define recursively on the formulas of set theory

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Let M be a class and E be a binary relation on M . We define recursively on the formulas of set theory

$$(x = y)^{M,E} =_{\text{df}} (x = y)$$

$$(x \in y)^{M,E} =_{\text{df}} \varphi_E(x, y)$$

$$(\neg \alpha)^{M,E} =_{\text{df}} \neg \alpha^{M,E}$$

$$(\alpha \rightarrow \beta)^{M,E} =_{\text{df}} \alpha^M \rightarrow \beta^{M,E}$$

$$(\forall x \alpha)^{M,E} =_{\text{df}} \forall x (\varphi_M(x) \rightarrow \alpha^{M,E})$$

Class model

Convention

- In case $E = \in \cap M$, we write φ^M instead of $\varphi^{M, \in \cap M}$
- We say E is **well-founded** on M if for each non-empty subset A of M there is an E -minimal element $a \in A$
- We say E is **set-like** on M if for each $a \in M$, $\{b \in M \mid b E a\}$ is a set
- We say M is **transitive** if for each $a \in M$, $a \subset M$

Class model

Theorem (Mostowski collapse)

Let E be well-founded and set-like on M , then we can recursively define a transitive class N such that $(M, E) \simeq (N, \in)$

In this case, we say N is the **Mostowski collapse** of (M, E)

Class model

Recall:

Lemma (Soundness)

$\sigma \vdash \delta$ implies $\sigma \models \delta$, i.e. for each structure \mathfrak{A} , if $\mathfrak{A} \models \sigma$, then $\mathfrak{A} \models \delta$

By a similar argument:

Lemma (PA or ZF – Inf)

If $\sigma \vdash \delta$, then for any formula φ_M, φ_E , $\sigma^{M,E} \vdash \delta^{M,E}$

Class model

To show $\text{Con}(\text{ZFC}^-) \rightarrow \text{Con}(\text{ZFC})$, we prove finitistically that

$$\text{ZFC}^- \vdash \sigma^{\text{WF}}$$

for each σ in ZFC

And then $\text{ZFC} \vdash x \neq x$ implies $\text{ZFC}^- \vdash (x \neq x)^{\text{WF}}$

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Class model

Lemma

For any class M ,

- If M is transitive, then Ext^M
- If $M \subset \text{WF}$, then Foundation^M
- If $\forall z \in M \forall y \subset z (y \in M)$, then Separation^M
- If $\forall x, y \in M (\{x, y\} \in M)$, then Pair^M
- If $\forall X \in M (\bigcup X \in M)$, then Union^M
- If M is transitive and for all function f , $\text{dom } f \in M$ and $\text{ran } f \subset M$ imply $\text{ran } f \in M$, then Rep^M

Class model

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- If $\forall z \in M \forall y \subset z (y \in M)$, then Separation^M
- If $\forall x, y \in M (\{x, y\} \in M)$, then Pair^M EXE:
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Class model

Lemma

Let M be a transitive class. Then

$$\forall x \in M \left((P(x) \cap M) \in M \right) \rightarrow \text{Pow}^M$$

Next on Set Theory

- Absoluteness