

# Set Theory II

## 集合论 II

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

Spring 2016

# Previously on Set Theory

- $\text{HF} = V_\omega$  (axiom of foundation)
- $\text{HF} \models \text{ZF} - \text{Inf} + \neg\text{Inf}$
- $\text{ZF} \vdash \text{Con}(\text{ZF} - \text{Inf} + \neg\text{Inf})$

$\downarrow$   
 $\text{Inf}$

EXE:  $ZF +$  there exists an inaccessible cardinal  $\vdash \text{Con}(ZF)$

## Definition

Let  $\Gamma, \Lambda$  be two sets of sentences in the language of set theory extending  $ZF - \text{Inf} - \text{Pow}$ . We say  $\Gamma$  is **proof-theoretically strictly stronger** than  $\Lambda$  (written  $\Gamma \triangleright \Lambda$ ) iff  $\Gamma \vdash \text{Con}(\Lambda)$

## Example

- $ZF \triangleright (ZF - \text{Inf} + \neg \text{Inf})$
- $(ZF + \text{there exists an inaccessible cardinal}) \triangleright ZF$
- $ZFC \triangleright (ZFC - \text{Pow})$

EXE: Define **hereditarily countable set** HC within ZFC,  
and show that  $HC \models (ZFC - \text{Pow})$

## Corollary

▷ is not reflexive

## Definition

We define  $\Gamma \leq \Lambda$  iff we have a **finitistic proof** that  $\text{Con}(\Lambda) \rightarrow \text{Con}(\Gamma)$ . We say  $\Lambda$  and  $\Gamma$  are proof-theoretically equivalent (written  $\Gamma \sim \Lambda$ ) iff  $\Gamma \leq \Lambda$  and  $\Lambda \leq \Gamma$

Clearly,  $\leq$  is transitive and reflexive

## Corollary

▷ is not reflexive

## Definition

PA, ZF - Int

We define  $\Gamma \leq \Lambda$  iff we have a **finitistic proof** that

$\text{Con}(\Lambda) \rightarrow \text{Con}(\Gamma)$ . We say  $\Lambda$  and  $\Gamma$  are **proof-theoretically equivalent** (written  $\Gamma \sim \Lambda$ ) iff  $\Gamma \leq \Lambda$  and  $\Lambda \leq \Gamma$

Clearly,  $\leq$  is transitive and reflexive

## Corollary

▷ is not reflexive

## Definition

We define  $\Gamma \leq \Lambda$  iff we have a **finitistic proof** that  $\text{Con}(\Lambda) \rightarrow \text{Con}(\Gamma)$ . We say  $\Lambda$  and  $\Gamma$  are **proof-theoretically equivalent** (written  $\Gamma \sim \Lambda$ ) iff  $\Gamma \leq \Lambda$  and  $\Lambda \leq \Gamma$

Clearly,  $\leq$  is transitive and reflexive

## Example (To be proved)

■  $PA \sim (ZF - Inf)$

■  $ZF^- \sim ZF \sim ZFC \sim (ZFC + CH) \sim (ZFC + \neg CH)$

$\overline{ZF^-}$  Foundation

$\overline{ZFC}$  =  $\overline{CH}$



## Lemma

$$\blacksquare \Gamma \subset \Lambda \xrightarrow{=} \Gamma \leq \Lambda$$

$$\Rightarrow \tau \leq \alpha$$

$$\blacksquare \Gamma \subset \Lambda \wedge \Lambda \leq \Gamma \rightarrow \Gamma \sim \Lambda$$

$$\Delta \vdash \text{Con}(\tau)$$

$$\text{Con}(\alpha) \rightarrow \text{Con}(\tau)$$

$$\blacksquare \Gamma \triangleleft \Lambda \rightarrow \underline{\Gamma \leq \Lambda}$$

$$\blacksquare \Gamma \leq \Lambda \wedge \Lambda \triangleleft \Theta \rightarrow \Gamma \triangleleft \Theta$$

■  $\triangleleft$  is transitive

Proof Assume  $\neg \text{Con}(\tau)$

is.  $\exists \alpha \tau \vdash \alpha \wedge \alpha \leq \tau$   $\Sigma_1$

Since  $\Delta \geq \exists \bar{\tau}$ -int (PA)

so  $\Delta \vdash \neg \text{Con}(\tau)$

And  $\Delta \vdash \text{Con}(\tau)$ , so

$\neg \text{Con}(\Delta)$

Gödel's completeness theorem says: if  $\Gamma \triangleleft \Lambda$  is provable, then you can always prove it by finding a set model  $M$  of  $\Gamma$  working in a model of  $\Lambda$

While Gödel's incompleteness theorem says: If you want to prove  $\Gamma \leq \Lambda$  with  $\Lambda \subset \Gamma$ , working in a universe of  $\Lambda$ , you should **not** find a set model of  $\Gamma$

Gödel's completeness theorem says: if  $\Gamma \triangleleft \Lambda$  is provable, then you can always prove it by finding a set model  $M$  of  $\Gamma$  working in a model of  $\Lambda$

While Gödel's incompleteness theorem says: If you want to prove  $\Gamma \leq \Lambda$  with  $\Lambda \subset \Gamma$ , working in a universe of  $\Lambda$ , you should **not** find a set model of  $\Gamma$

# Class model

Recall: In metalanguage, A **class**  $A$  is always of the form  $\{x \mid \varphi_A(x)\}$  for some formula  $\varphi_A$  in the language of set theory

## Example (proper class)

- $V = \{x \mid x = x\}$
- $\in = \{(x, y) \mid x \in y\}$
- $\text{OR} = \{\alpha \mid \alpha \text{ is transitive and well ordered by } \in\}$
- EXE:  $\text{WF} = \bigcup_{\alpha \in \text{OR}} V_\alpha = ?$

# Class model

To prove:  $\text{Con}(\text{ZFC}^-) \rightarrow \text{Con}(\text{ZFC})$

Intuitively, we assume  $V \models \text{ZFC}^-$ , and show  $WF \models \text{ZFC}$

But we cannot even say  $V \models \text{ZFC}^-$  or  $WF \models \text{ZFC}$  in set theory

# Class model

To prove:  $\text{Con}(\text{ZFC}^-) \rightarrow \text{Con}(\text{ZFC})$

Intuitively, we assume  $V \models \text{ZFC}^-$ , and show  $WF \models \text{ZFC}$

But we cannot even say  $V \models \text{ZFC}^-$  or  $WF \models \text{ZFC}$  in set theory

# Class model

To prove:  $\text{Con}(\text{ZFC}^-) \rightarrow \text{Con}(\text{ZFC})$

Intuitively, we assume  $V \models \text{ZFC}^-$ , and show  $WF \models \text{ZFC}$

But we cannot even say  $V \models \text{ZFC}^-$  or  $WF \models \text{ZFC}$  in set theory

# Class model

Definition (in meta theory)

Let  $M = \{x \mid \varphi_M(x)\}$  be a class and

$E = \{(x, y) \mid \varphi_E(x, y) \wedge \varphi_M(x) \wedge \varphi_M(y)\}$  be a binary relation on

$M$ . We define recursively on the formulas of set theory



# Class model

Definition (in meta theory)

Let  $M$  be a class and  $E$  be a binary relation on  $M$ . We define recursively on the formulas of set theory

# Class model

## Definition (in meta theory)

Let  $M$  be a class and  $E$  be a binary relation on  $M$ . We define recursively on the formulas of set theory

$$(x = y)^{M,E} =_{\text{df}} (x = y)$$

$$(x \in y)^{M,E} =_{\text{df}} \varphi_E(x, y)$$

$$(\neg \alpha)^{M,E} =_{\text{df}} \neg \alpha^{M,E}$$

$$(\alpha \rightarrow \beta)^{M,E} =_{\text{df}} \alpha^M \rightarrow \beta^{M,E}$$

$$(\forall x \alpha)^{M,E} =_{\text{df}} \forall x (\varphi_M(x) \rightarrow \alpha^{M,E})$$

# Class model

## Convention

- In case  $E = \in \cap M$ , we write  $\varphi^M$  instead of  $\varphi^{M, \in \cap M}$
- We say  $E$  is **well-founded** on  $M$  if for each non-empty subset  $A$  of  $M$  there is an  $E$ -minimal element  $a \in A$
- We say  $E$  is **set-like** on  $M$  if for each  $a \in M$ ,  
 $\{b \in M \mid b E a\}$  is a set
- We say  $M$  is **transitive** if for each  $a \in M$ ,  $a \subset M$

# Class model

## Theorem (Mostowski collapse)

Let  $E$  be well-founded and set-like on  $M$ , then we can recursively define a transitive class  $N$  such that  $(M, E) \simeq (N, \in)$

In this case, we say  $N$  is the **Mostowski collapse** of  $(M, E)$

# Class model

Recall:

## Lemma (Soundness)

$\sigma \vdash \delta$  implies  $\sigma \models \delta$ , i.e. for each structure  $\mathfrak{A}$ , if  $\mathfrak{A} \models \sigma$ , then  $\mathfrak{A} \models \delta$

By a similar argument:

## Lemma (PA or ZF – Inf )

If  $\sigma \vdash \delta$ , then for any formula  $\varphi_M, \varphi_E$ ,  $\sigma^{M,E} \vdash \delta^{M,E}$

# Class model

To show  $\text{Con}(\text{ZFC}^-) \rightarrow \text{Con}(\text{ZFC})$ , we prove finitistically that

$$\text{ZFC}^- \vdash \sigma^{\text{WF}}$$

for each  $\sigma$  in ZFC

And then  $\text{ZFC} \vdash x \neq x$  implies  $\text{ZFC}^- \vdash (x \neq x)^{\text{WF}}$

# Class model

To show  $\text{Con}(\text{ZFC}^-) \rightarrow \text{Con}(\text{ZFC})$ , we prove finitistically that

$$\text{ZFC}^- \vdash \sigma^{\text{WF}}$$

for each  $\sigma$  in ZFC

And then  $\text{ZFC} \vdash x \neq x$  implies  $\text{ZFC}^- \vdash (x \neq x)^{\text{WF}}$

# Class model

## Lemma

For any class  $M$ ,

- If  $M$  is transitive, then  $\text{Ext}^M$
- If  $M \subset \text{WF}$ , then  $\text{Foundation}^M$
- If  $\forall z \in M \forall y \subset z (y \in M)$ , then  $\text{Separation}^M$
- If  $\forall x, y \in M (\{x, y\} \in M)$ , then  $\text{Pair}^M$
- If  $\forall X \in M (\cup X \in M)$ , then  $\text{Union}^M$
- If  $M$  is transitive and for all function  $f$ ,  $\text{dom } f \in M$  and  $\text{ran } f \subset M$  imply  $\text{ran } f \in M$ , then  $\text{Rep}^M$



# Class model

## Lemma

For any class  $M$ ,

- If  $M$  is transitive, then  $\text{Ext}^M$
- If  $M \subset \text{WF}$ , then  $\text{Foundation}^M$  EXE:
- If  $\forall z \in M \forall y \subset z (y \in M)$ , then  $\text{Separation}^M$
- If  $\forall x, y \in M (\{x, y\} \in M)$ , then  $\text{Pair}^M$  EXE:
- If  $\forall X \in M (\cup X \in M)$ , then  $\text{Union}^M$  EXE:
- If  $M$  is transitive and for all function  $f$ ,  $\text{dom } f \in M$  and  $\text{ran } f \subset M$  imply  $\text{ran } f \in M$ , then  $\text{Rep}^M$

# Class model

## Lemma

Let  $M$  be a transitive class. Then

$$\forall x \in M \left( (P(x) \cap M) \in M \right) \rightarrow \text{Pow}^M$$

# Next on Set Theory

- Absoluteness