Set Theory II

集合论 II

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

Spring 2016

Previously on Set Theory

On 1st-order logic

- Language: symbols, terms, formulas
- Semantics: structure, satisfactory, truth
- Proofs: Axioms, rules, proof (deduction)

Previously on Set Theory

Theorem (Gödel's completeness theorem)

$$\Sigma \vDash \varphi \Leftrightarrow \Sigma \vdash \varphi$$

Previously on Set Theory

Theorem (Gödel's completeness theorem (ver. 2))

Σ is consistent $\Leftrightarrow$ Σ is satisfiable

Previously on Set Theory

On set theory

- Axiomatic set theory, say ZF is formalized in the 1st-order language $\{=, \in\}$
- ZF is constituted by eight groups of axioms

Previously on Set Theory

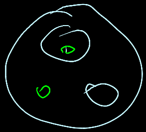
$x_0 \ni x_1 \ni x_2 \ni x_3 \cdots$

$x \in x$, $\{x\}$

On set theory

$\forall x \forall y \, (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$

- Axiom of extensionality
- Axiom of foundation



$\forall x \, (x \neq \emptyset \rightarrow \exists y \, (y \in x \wedge \forall z \, (z \in x \rightarrow z \notin y)))$

Previously on Set Theory

On set theory

$\forall x \forall y \exists z (\forall w (w \in z \leftrightarrow (w = x \vee w = y)))$

- Axiom of pair

  $\forall x \forall y \exists z (z = \{x, y\})$

- Axiom of union

  $\forall X \exists Y (\forall z (z \in Y \leftrightarrow \exists w (w \in X \wedge z \in w)))$

  $\forall x \exists Y (Y = \bigcup X)$

- Axiom of power set

  $\forall w (w \in z \rightarrow w \in X)$

  $\forall X \exists Y (\forall z (z \in Y \leftrightarrow z \subset X))$

  $\forall x \exists Y (Y = \mathcal{P}(x))$

Previously on Set Theory

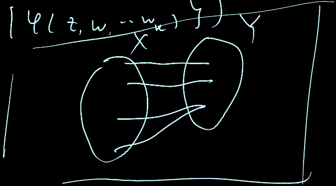
Fix $\varphi(x)$

$$\forall X \exists Y (\forall z (z \in Y \leftrightarrow z \in X \wedge \varphi(z)))$$

$$\forall X \exists Y (Y = \{ z \in X \mid \varphi(z) \})$$

$$\forall w_1 \cdots \forall w_k \forall X \exists Y (Y = \{ z \in X \mid \varphi(z, w_1, \ldots, w_k) \})$$

On set theory



- Separation schema
- Replacement Schema

Fix $\varphi(x, y)$

$$\forall X [\forall x \in X \exists ! y \, \varphi(x,y) \rightarrow \exists Y \forall x \in X \exists y \in Y \, \varphi(x,y)]$$

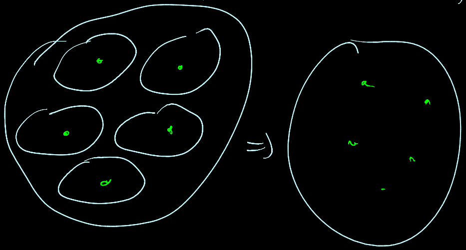
Previously on Set Theory

$\phi \in X \wedge \forall z (z \in X \to z \cup \{z\} \in X)$

On set theory $\quad \exists x \, (x \text{ is inductive})$

- Axiom of infinite (ZF so far) $\quad \{\phi, \{\phi\}, \{\phi, \{\phi\}\}, \ldots\}$
- Axiom of choice

ZFC is ZF plus AC.



A “toy model” for ZF− Infinity

V_ω

1 2 3

0, {0}, {0, 1}, {0, 1, 2}

Recall: Informally $\mathbb{N} = \omega = \Big\{\emptyset, \{\emptyset\}, \big\{\emptyset, \{\emptyset\}\big\}, \ldots\Big\}$

EXE: Provide formal definition of ω

Definition

- $R(0) = V_0 = \emptyset$
- $R(n+1) = V_{n+1} = P(V_n)$ for all $n \in \omega$
- $R(\omega) = V_\omega = \bigcup\{V_n \mid n \in \omega\}$

EXE: Show that V_ω exists?

V_ω

$X = \omega \leftrightarrow X$ is inductive $\wedge \; \forall Y\,(Y$ is inductive $\to X \subset Y)$

Recall: Informally $\mathbb{N} = \omega = \Big\{\emptyset, \{\emptyset\}, \big\{\emptyset, \{\emptyset\}\big\}, \ldots\Big\}$

EXE: Provide formal definition of ω

Definition

- $R(0) = V_0 = \emptyset$
- $R(n+1) = V_{n+1} = P(V_n)$ for all $n \in \omega$
- $R(\omega) = V_\omega = \bigcup\{V_n \mid n \in \omega\}$

EXE: Show that V_ω exists?

Recall: Informally $\mathbb{N} = \omega = \Big\{\emptyset, \{\emptyset\}, \big\{\emptyset, \{\emptyset\}\big\}, \dots\Big\}$

EXE: Provide formal definition of ω

Definition

- $R(0) = V_0 = \emptyset$
- $R(n+1) = V_{n+1} = P(V_n)$ for all $n \in \omega$
- $R(\omega) = V_\omega = \bigcup\{V_n \mid n \in \omega\}$

EXE: Show that V_ω exists?

V_ω

Recall: Informally $\mathbb{N} = \omega = \Big\{\emptyset, \{\emptyset\}, \big\{\emptyset, \{\emptyset\}\big\}, \ldots\Big\}$

EXE: Provide formal definition of ω

Definition

- $R(0) = V_0 = \emptyset$
- $R(n+1) = V_{n+1} = P(V_n)$ for all $n \in \omega$
- $R(\omega) = V_\omega = \bigcup\{V_n \mid n \in \omega\}$

EXE: Show that V_ω exists?

Hereditarily finite sets

A set X is finite if there exists $n < \omega$ and a bijection $f : n \to X$

i.e. $X = \{x_0, \ldots, x_{n-1}\}$ if $n \geq 1$ and $x_i = f(i)$ for $0 \leq i \leq 1$

Hereditarily finite sets

A set X is finite if there exists $n < \omega$ and a bijection $f\colon n \to X$

i.e. $X = \{x_0, \ldots, x_{n-1}\}$ if $n \geq 1$ and $x_i = f(i)$ for $0 \leq i \leq 1$

Hereditarily finite sets

A set Y is transitive if for each $x \in Y$, $x \subset Y$

The transitive closure of X (written $\mathrm{TC}(X)$) is the smallest transitive set containing X as an element

EXE V_n is transitive $(n < \omega)$

Hereditarily finite sets

Definition

Fix a set X, define recursively

- $U_0(X) = X$
- $U_{n+1}(X) = \bigcup U_n(X)$

Fact

$$\mathrm{TC}(X) = \{X\} \cup \bigcup_{n \in \omega} U_n(X)$$

Hereditarily finite sets

Definition

Fix a set X, define recursively

- $U_0(X) = X$
- $U_{n+1}(X) = \bigcup U_n(X)$

Fact

$$\mathrm{TC}(X) = \{X\} \cup \bigcup_{n \in \omega} U_n(X)$$

Hereditarily finite sets

A set X is hereditarily finite if $\mathsf{TC}(X)$ is finite. Let HF be the class of all hereditarily finite sets

Lemma

$\mathsf{HF} = V_\omega$

To show $V_\omega \subseteq HF$:

We prove by induction that $V_n \in HF$ for each $n \in \mathbb{N}$

Claim If $|TC(V_n)| = k+1$ then $|TC(V_{n+1})| = 2^k + 1$

Hereditarily finite sets

A set X is hereditarily finite if $\mathrm{TC}(X)$ is finite. Let HF be the class of all hereditarily finite sets

Lemma

$\mathrm{HF} = V_\omega$

To show $HF \subseteq V_\omega$:

Claim Assume $x \in HF$, then there is a least $n \in \omega$ s.t.

$U_n(x) = \phi$, i.e. there is no sequence on length $n+1$ in $TC(x)$ as follow

$x \ni x_1 \ni x_2 \cdots x_{n+1}$

Claim in this case, $x \in V_{n+1}$

What's in HF?

Everything but infinities

Everything but infinities

What's in HF?

Fact

- $\emptyset \in HF$
- If $x, y \in \mathrm{HF}$, then $\{x, y\} \in \mathrm{HF}$

Corollary

$(\mathrm{HF}, \in \restriction \mathrm{HF}) \models$ axiom of pair

What's in HF?

Fact

- $\emptyset \in HF$
- If $x, y \in \mathrm{HF}$, then $\{x, y\} \in \mathrm{HF}$

Corollary

$(\mathrm{HF}, \in\restriction\mathrm{HF}) \models$ axiom of pair

What's in HF?

Fact

- $\emptyset \in HF$
- If $x, y \in$ HF, then $\{x, y\} \in$ HF

Corollary

HF $\models$ axiom of pair

What's in HF?

Fact

- If $X \in \mathrm{HF}$, then $\bigcup X \in \mathrm{HF}$
- If $x, y \in \mathrm{HF}$, then $x \cup y \in \mathrm{HF}$

Corollary

$\mathrm{HF} \vDash$ axiom of union

What's in HF?

Fact

- If $X \in \mathrm{HF}$ and $Y \subset \mathrm{HF}$, then $Y \in \mathrm{HF}$
- If $X \in \mathrm{HF}$, then $P(X) \in \mathrm{HF}$

Corollary

- $\mathrm{HF} \vDash$ separation schema
- $\mathrm{HF} \vDash$ axiom of power set

What's in HF?

Recall: the successor of x, $Sx = x \cup \{x\}$

Fact

- If $x \in \mathrm{HF}$, then $Sx \in \mathrm{HF}$
- $\omega \subset \mathrm{HF}$

What's in HF?

Recall: the ordered pair (x, y) is defined to be $\{\{x\}, \{x, y\}\}$

EXE: If $x, y \in \mathrm{HF}$, then $(x, y) \in \mathrm{HF}$

What's in HF?

Recall: the ordered pair (x, y) is defined to be $\big\{\{x\}, \{x, y\}\big\}$

EXE: If $x, y \in \mathrm{HF}$, then $(x, y) \in \mathrm{HF}$

What's in HF?

EXE: Give a definition of integer and rational number such that the set of all integers $\mathbb{Z}$ and the set of all rational number $\mathbb{Q}$ are both included in HF

What's in HF?

Recall:

- $X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$
- A relation on $X \times Y$ is a subset of $X \times Y$
- A function $f : X \to Y$ is a relation on $X \times Y$ such that for each $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in f$

What's in HF?

Fact

Fix $X, Y \in \mathrm{HF}$, then $X \times Y \in \mathrm{HF}$ and every relation / function on $X \times Y$ is in HF

Recall: a finite sequence of elements in X is a injection $s : n \to X$

Fact

If $X \subset \mathrm{HF}$, then any finite sequence of elements in X is in HF

What's in HF?

Recall: By Gödel numbering, we can define symbols of a 1st-order language to be natural numbers, and so sets in HF

EXE: Show that under Gödel numbering, all terms, formulas, and even proofs are in HF

What's in HF?

Fact

- HF $\models$ replacement schema
- HF $\models$ axiom of choice

Corollary

- HF $\models$ ZFC$-$ axiom of infinite
- ZFC minus axiom of infinite is consistent

Next on Set Theory

- models of set theory
- consistency proof
- absoluteness