

Set Theory II

集合论 II

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Previously on Set Theory

On 1st-order logic

- Language: symbols, terms, formulas
- Semantics: structure, satisfactory, truth
- Proofs: Axioms, rules, proof (deduction)

Previously on Set Theory

Theorem (Gödel's completeness theorem)

$$\Sigma \models \varphi \Leftrightarrow \Sigma \vdash \varphi$$

Previously on Set Theory

Theorem (Gödel's completeness theorem (ver. 2))

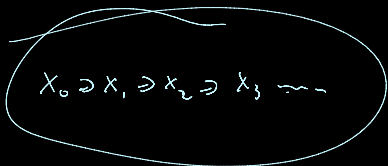
Σ is consistent $\Leftrightarrow \Sigma$ is satisfiable

Previously on Set Theory

On set theory

- Axiomatic set theory, say ZF is formalized in the 1st-order language $\{=, \in\}$
- ZF is constituted by eight groups of axioms

Previously on Set Theory



On set theory $\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$

- Axiom of extensionality
- Axiom of foundation



$$\forall x (x \neq \emptyset \rightarrow \exists y (y \in x \wedge \forall z (z \in y \rightarrow z \neq x)))$$

Previously on Set Theory

On set theory $\forall x \forall y \exists z (\forall w (w \in z \leftrightarrow (w = x \vee w = y)))$

- Axiom of pair $\forall x \forall y \exists z (z = \{x, y\})$

- Axiom of union $\forall x \exists y (\forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w)))$
 $(X \cup = Y) \forall z (z \in Y \leftrightarrow z \in X)$

- Axiom of power set $\forall x \exists y (\forall z (z \in y \leftrightarrow z \subset x))$
 $(X \subset y \rightarrow z \in y) \forall w (w \in z \rightarrow w \in X)$
 $(\mathcal{P}(X) = Y) \forall z (z \in Y \leftrightarrow z \subset X)$

Previously on Set Theory

$$\underline{\text{Fix } \varphi(x)}$$

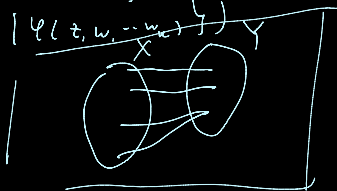
$$\forall X \exists Y (\forall z (z \in Y \leftrightarrow z \in X \wedge \varphi(z)))$$

$$\forall X \exists Y (Y = \{ z \in X \mid \varphi(z) \})$$

$$\forall w_1, \dots, \forall w_n \forall X \exists Y (Y = \{ z \in X \mid \varphi(z, w_1, \dots, w_n) \})$$

On set theory

- Separation schema
- Replacement Schema



$$\text{Fix } \varphi(x, y)$$

$$\forall X [\forall x \in X \exists ! y \varphi(x, y) \rightarrow \exists Y \forall x \in X \exists y \in Y \varphi(x, y)]$$

Previously on Set Theory

$$\phi \in X \wedge \forall z (z \in X \rightarrow z \cup \{z\} \in X)$$

On set theory

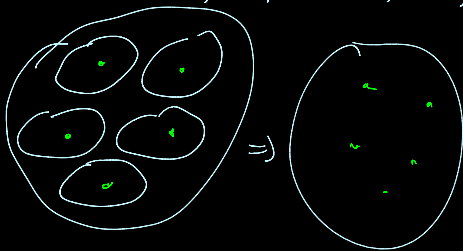
$\exists x (x \text{ is inductive})$

■ Axiom of infinite (ZF so far)

$\{ \emptyset, \{ \emptyset \}, \{ \emptyset, \{ \emptyset \} \}, \dots \}$

■ Axiom of choice

ZFC is ZF plus AC.



A “toy model” for ZF– Infinity

$$V_\omega$$

$$\{0, \{0\}, \{0, \{0, 1\}\}, \dots\}$$

1 2 3

Recall: Informally $\mathbb{N} = \omega = \{0, \{0\}, \{0, \{0\}\}, \dots\}$

EXE: Provide formal definition of ω

Definition

- $R(0) = V_0 = \emptyset$
- $R(n+1) = V_{n+1} = P(V_n)$ for all $n \in \omega$
- $R(\omega) = V_\omega = \bigcup \{V_n \mid n \in \omega\}$

EXE: Show that V_ω exists?

$$X = \omega \iff \underbrace{X \text{ is inductive}} \wedge \forall Y (Y \text{ is inductive} \rightarrow X \subset Y) \quad V_\omega$$

Recall: Informally $\mathbb{N} = \omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \dots\}$

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Hereditarily finite sets

A set X is **finite** if there exists $n < \omega$ and a bijection $f: n \rightarrow X$

i.e. $X = \{x_0, \dots, x_{n-1}\}$ if $n \geq 1$ and $x_i = f(i)$ for $0 \leq i < n$

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Hereditarily finite sets

A set Y is **transitive** if for each $x \in Y$, $x \subset Y$

The **transitive closure** of X (written $\text{TC}(X)$) is the smallest transitive set containing X as an element

EXE \forall_n is transitive ($n \in \omega$)

Hereditarily finite sets

Definition

Fix a set X , define recursively

- $U_0(X) = X$
- $U_{n+1}(X) = \cup U_n(X)$

Fact

$$\text{TC}(X) = \{X\} \cup \bigcup_{n \in \omega} U_n(X)$$

Hereditarily finite sets

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Fix a set X , define recursively

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Fact

$$\text{TC}(X) = \{X\} \cup \bigcup_{n \in \omega} U_n(X)$$

Hereditarily finite sets

A set X is **hereditarily finite** if $\text{TC}(X)$ is finite. Let **HF** be the **class** of all hereditarily finite sets

Lemma

$$\text{HF} = V_\omega$$

$\forall x (TC(x) \text{ is finite} \leftrightarrow \exists n x \in V_n)$ **Hereditarily finite sets**

(\Leftarrow) for all n , $V_n \in HF$ $\left\{ \begin{array}{l} \text{ii) Assume } V_n \in HF, \text{ i.e. } TC(V_n) = \{V_n\} \cup V_n \\ \text{By induction} \\ \text{ii) } V_0 = \emptyset \in HF \end{array} \right.$

A set X is **hereditarily finite** if $TC(X)$ is finite. Let **HF** be the **class** of all hereditarily finite sets

Lemma

$HF = V_\omega$

$$\begin{aligned} \text{ii) } TC(V_0) &= \{V_0\} \cup \underbrace{V_0}_{\emptyset} \cup \underbrace{V_0}_{\emptyset} \\ &= \{V_0\} \cup V_0 \end{aligned}$$

$$\text{iii) } TC(V_{n+1}) = \{V_{n+1}\} \cup V_{n+1} \cup \dots$$

What's in HF?

Everything but infinities

Everything but infinities

What's in HF?

Fact

- $\emptyset \in HF$
- If $x, y \in HF$, then $\{x, y\} \in HF$

Corollary

$(HF, \in \upharpoonright HF) \models$ axiom of pair

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HF \models axiom of pair

What's in HF?

Fact

- If $X \in \text{HF}$, then $\bigcup X \in \text{HF}$
- If $x, y \in \text{HF}$, then $x \cup y \in \text{HF}$

Corollary

HF \models axiom of union

What's in HF?

Fact

- If $X \in \text{HF}$ and $Y \subset \text{HF}$, then $Y \in \text{HF}$
- If $X \in \text{HF}$, then $P(X) \in \text{HF}$

Corollary

- $\text{HF} \models$ separation schema
- $\text{HF} \models$ axiom of power set

What's in HF?

Recall: the **successor** of x , $Sx = x \cup \{x\}$

Fact

- If $x \in \text{HF}$, then $Sx \in \text{HF}$
- $\omega \subset \text{HF}$

What's in HF?

Recall: the **ordered pair** (x, y) is defined to be $\{\{x\}, \{x, y\}\}$

EXE: If $x, y \in HF$, then $(x, y) \in HF$

What's in HF?

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EXE: If $x, y \in HF$, then $(x, y) \in HF$

What's in HF?

EXE: Give a definition of **integer** and **rational number** such that the set of all integers \mathbb{Z} and the set of all rational number \mathbb{Q} are both included in HF

What's in HF?

Recall:

- $X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$
- A relation on $X \times Y$ is a subset of $X \times Y$
- A function $f: X \rightarrow Y$ is a relation on $X \times Y$ such that for each $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in f$

What's in HF?

Fact

Fix $X, Y \in \text{HF}$, then $X \times Y \in \text{HF}$ and every relation / function on $X \times Y$ is in HF

Recall: a **finite sequence** of elements in X is a injection

$$s : n \rightarrow X$$

Fact

If $X \subset \text{HF}$, then any finite sequence of elements in X is in HF

What's in HF?

Recall: By Gödel numbering, we can define symbols of a 1st-order language to be natural numbers, and so sets in HF

EXE: Show that under Gödel numbering, all terms, formulas, and even proofs are in HF

What's in HF?

Fact

- $\text{HF} \models$ replacement schema
- $\text{HF} \models$ axiom of choice

Corollary

- $\text{HF} \models \text{ZFC} -$ axiom of infinite
- ZFC minus axiom of infinite is consistent

Next on Set Theory

- models of set theory
- consistency proof
- absoluteness