

# Set Theory II

## 集合论 II

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# Previously on Set Theory

## On 1st-order logic

- Language: symbols, terms, formulas
- Semantics: structure, satisfactory, truth
- Proofs: Axioms, rules, proof (deduction)

# Previously on Set Theory

Theorem (Gödel's completeness theorem)

$$\Sigma \models \varphi \Leftrightarrow \Sigma \vdash \varphi$$

# Previously on Set Theory

Theorem (Gödel's completeness theorem (ver. 2))

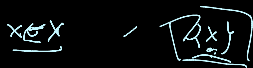
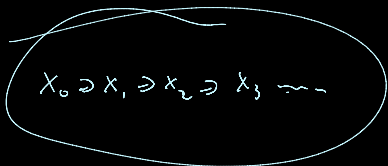
$\Sigma$  is consistent  $\Leftrightarrow \Sigma$  is satisfiable

# Previously on Set Theory

On set theory

- Axiomatic set theory, say ZF is formalized in the 1st-order language  $\{=, \in\}$
- ZF is constituted by eight groups of axioms

# Previously on Set Theory



On set theory  $\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$

- Axiom of extensionality
- Axiom of foundation



$$\forall x (x \neq \emptyset \rightarrow \exists y (y \in x \wedge \forall z (z \in y \rightarrow z \neq x)))$$

# Previously on Set Theory

On set theory  $\forall x \forall y \exists z ( \forall w ( w \in z \Leftrightarrow (w = x \vee w = y) ) )$   
 $\forall x \forall y \exists z ( z )$

- Axiom of pair
- Axiom of union
- Axiom of power set

# Previously on Set Theory

On set theory

- Separation schema
- Replacement Schema



# Previously on Set Theory

On set theory

- Axiom of infinite (ZF so far)
- Axiom of choice

ZFC is ZF plus AC.

A “toy model” for ZF– Infinity

$V_\omega$ 

Recall: Informally  $\mathbb{N} = \omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \dots\}$

EXE: Provide formal definition of  $\omega$

Definition

- $R(0) = V_0 = \emptyset$
- $R(n+1) = V_{n+1} = P(V_n)$  for all  $n \in \omega$
- $R(\omega) = V_\omega = \bigcup \{V_n \mid n \in \omega\}$

EXE: Show that  $V_\omega$  exists?

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# Hereditarily finite sets

A set  $X$  is **finite** if there exists  $n < \omega$  and a bijection  $f: n \rightarrow X$

i.e.  $X = \{x_0, \dots, x_{n-1}\}$  if  $n \geq 1$  and  $x_i = f(i)$  for  $0 \leq i < n$

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# Hereditarily finite sets

A set  $Y$  is **transitive** if for each  $x \in Y$ ,  $x \subset Y$

The **transitive closure** of  $X$  (written  $TC(X)$ ) is the smallest transitive set containing  $X$  as an element

# Hereditarily finite sets

## Definition

Fix a set  $X$ , define recursively

- $U_0(X) = X$
- $U_{n+1}(X) = \cup U_n(X)$

## Fact

$$\text{TC}(X) = \{X\} \cup \bigcup_{n \in \omega} U_n(X)$$

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# Hereditarily finite sets

A set  $X$  is **hereditarily finite** if  $\text{TC}(X)$  is finite. Let **HF** be the **class** of all hereditarily finite sets

Lemma

$$\text{HF} = V_\omega$$

# Hereditarily finite sets

A set  $X$  is **hereditarily finite** if  $\text{TC}(X)$  is finite. Let **HF** be the **class** of all hereditarily finite sets

Lemma

$$\text{HF} = V_\omega$$

What's in HF?

Everything but infinities

Everything but infinities



# What's in HF?

## Fact

- $\emptyset \in HF$
- If  $x, y \in HF$ , then  $\{x, y\} \in HF$

## Corollary

$(HF, \in \upharpoonright HF) \models$  axiom of pair

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**HF**  $\models$  axiom of pair

# What's in HF?

## Fact

- If  $X \in \text{HF}$ , then  $\bigcup X \in \text{HF}$
- If  $x, y \in \text{HF}$ , then  $x \cup y \in \text{HF}$

## Corollary

HF  $\models$  axiom of union

# What's in HF?

## Fact

- If  $X \in \text{HF}$  and  $Y \subset \text{HF}$ , then  $Y \in \text{HF}$
- If  $X \in \text{HF}$ , then  $P(X) \in \text{HF}$

## Corollary

- $\text{HF} \models$  separation schema
- $\text{HF} \models$  axiom of power set

# What's in HF?

Recall: the **successor** of  $x$ ,  $Sx = x \cup \{x\}$

Fact

- If  $x \in \text{HF}$ , then  $Sx \in \text{HF}$
- $\omega \subset \text{HF}$

# What's in HF?

Recall: the **ordered pair**  $(x, y)$  is defined to be  $\{\{x\}, \{x, y\}\}$

EXE: If  $x, y \in HF$ , then  $(x, y) \in HF$

# What's in HF?

Recall: the **ordered pair**  $(x, y)$  is defined to be  $\{\{x\}, \{x, y\}\}$

EXE: If  $x, y \in HF$ , then  $(x, y) \in HF$



# What's in HF?

EXE: Give a definition of **integer** and **rational number** such that the set of all integers  $\mathbb{Z}$  and the set of all rational number  $\mathbb{Q}$  are both included in HF

# What's in HF?

Recall:

- $X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$
- A relation on  $X \times Y$  is a subset of  $X \times Y$
- A function  $f: X \rightarrow Y$  is a relation on  $X \times Y$  such that for each  $x \in X$  there exists a unique  $y \in Y$  such that  $(x, y) \in f$

# What's in HF?

## Fact

Fix  $X, Y \in \text{HF}$ , then  $X \times Y \in \text{HF}$  and every relation / function on  $X \times Y$  is in HF

Recall: a **finite sequence** of elements in  $X$  is a injection

$$s: n \rightarrow X$$

## Fact

If  $X \subset \text{HF}$ , then any finite sequence of elements in  $X$  is in HF

# What's in HF?

Recall: By Gödel numbering, we can define symbols of a 1st-order language to be natural numbers, and so sets in HF

EXE: Show that under Gödel numbering, all terms, formulas, and even proofs are in HF

# What's in HF?

## Fact

- $\text{HF} \models$  replacement schema
- $\text{HF} \models$  axiom of choice

## Corollary

- $\text{HF} \models \text{ZFC} -$  axiom of infinite
- ZFC minus axiom of infinite is consistent

# Next on Set Theory

- models of set theory
- consistency proof
- absoluteness