

# Set Theory II

## 集合论 II

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Before we start



<http://logic.fudan.edu.cn/>

# 考核方式

平时

期末

# Before we start

## 参考书目：

- Kenneth Kunen. *Set Theory: Mathematical logic and Foundations*, College Publications, 2011.
- 郝兆宽、杨跃. *集合论：对无穷概念的探索*, 复旦大学出版社, 2014.
- Ralf Schindler. *Set Theory: Exploring Independence and Truth*, Springer, 2014.

It's all about **truth**

It's about mathematical truth

# mathematical truth

Two characterization of mathematical truth

- By proof
- By meaning

# mathematical truth

Two characterization of mathematical truth

- By *syntax*
- By meaning



# mathematical truth

Two characterization of mathematical truth

- By **syntax**
- By **semantics**

# mathematical truth

In either case, truths are true sentences

What is a sentence?

# mathematical truth

In either case, truths are true sentences

What is a **sentence**?

If it is one of the tasks of philosophy to break the domination of the word over the human spirit by laying bare the misconceptions that through the use of language often almost unavoidably arise concerning the relations between concepts and by **freeing thought from that with which only the means of expression of ordinary language**, constituted as they are, **saddle it**, then my **ideography**, further developed for these purposes, can **become a useful tool for the philosopher**. (*Begriffsschrift*)

# mathematicians' language

## Symbols

- (infinitely many) variables:  $v_0, v_1, v_2, \dots$
- connective:  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \dots$
- parentheses:  $(, )$
- predicates:  $=, P_0^{k_0}, P_1^{k_1}, P_2^{k_2}, \dots$
- function symbols:  $f_0^{k_0}, f_1^{k_1}, f_2^{k_2}, \dots$
- quantifiers:  $\forall, \exists$

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# mathematicians' language

## Example (language of arithmetic)

- predicates:  $=, \leq$
- function symbols:  $0, S, +, \cdot, E$

## Example (language of set theory)

- predicates:  $=, \in$
- no function symbols

# mathematicians' language

## Terms

- variables  $v_1, v_2, \dots$  are **terms**
- If  $f$  is a  $k$ -place function symbol, and  $t_1, \dots, t_k$  are terms, then  $ft_1 \dots t_k$  is a **term**
- no other terms

# mathematicians' language

## Formulas

- If  $P$  is a  $k$ -place function symbol, and  $t_1, \dots, t_k$  are terms, then  $Pt_1 \dots t_k$  is a **formula**
- If  $\varphi$  and  $\psi$  are formulas, then  $\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi, \varphi \leftrightarrow \psi, \dots$  are **formulas**
- If  $\varphi$  is a formula,  $x$  is a variable, then  $\forall x\varphi$  and  $\exists x\varphi$  are **formulas**
- no other formulas

# mathematicians' language

## Example

- language of arithmetic

$$\forall v_0 (v_0 \neq 0 \rightarrow \exists v_1 (v_0 = Sv_1))$$

- language of set theory

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y))$$

# mathematicians' language

## Example

- language of arithmetic

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An variable  $x$  **occurs freely** in  $\varphi$  if it is not **bounded by** any quantifiers

EXE: provide a definition

Convention

When we write  $\varphi(x_1, \dots, x_n)$ , we indicate there are **at most**  $x_1, \dots, x_n$  occur freely in  $\varphi$

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# mathematicians' language

**Sentences** are formulas with no variable occurring freely in it

When is a sentence said to be **true**?

- What is the meaning of a sentence?
- When is it provable?

# Semantics

## Interpretable symbols

- variables:  $v_0, v_1, v_2, \dots$
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- quantifiers:  $\forall, \exists$

# Semantics

Fix a language  $\mathcal{L}$ , a  $\mathcal{L}$ -structure is a function, which assigns an appropriate meaning to every interpretable symbols.

## Example

Intuitively, The standard structure of natural numbers

$\mathfrak{N} = (\mathbb{N}, \leq^{\mathfrak{N}}, 0^{\mathfrak{N}}, +^{\mathfrak{N}}, \cdot^{\mathfrak{N}}, E^{\mathfrak{N}})$  maps the domain  $(\forall, \exists)$  to the standard set of natural numbers  $\mathbb{N}$ ,  $\leq$  to the less or equal relation  $\leq^{\mathfrak{N}}$  on  $\mathbb{N}$ ,  $0$  the very natural number  $0^{\mathfrak{N}}$ ,  $+$  the addition operation  $+^{\mathfrak{N}}$  on  $\mathbb{N}$ , etc.

# Semantics

Fix a language  $\mathcal{L}$ , and a  $\mathcal{L}$ -structure  $\mathfrak{M}$ , an **assignment**  $s$  is a function, which assigns every variable an element in the **domain**  $M$  of the structure, i.e.  $s : V \rightarrow M$

# Semantics

Formal definition (Tarski's truth definition)

Fix a language  $\mathcal{L}$ , an  $\mathcal{L}$ -structure  $\mathfrak{M}$ , and an assignment  $s : V \rightarrow M$ , then we define  $\bar{s} : T \rightarrow M$ :

- $\bar{s}(v_i) = s(v_i)$
- $\bar{s}(ft_1 \dots t_k) = f^{\mathfrak{M}}(\bar{s}(t_1), \dots, \bar{s}(t_k))$

# Semantics

And we say

- $\mathfrak{M} \models Pt_1 \dots t_k[s]$  if  $(\bar{s}(t_1), \dots, \bar{s}(t_k)) \in P^{\mathfrak{M}}$
- $\mathfrak{M} \models \neg\varphi[s]$  if  $\mathfrak{M} \not\models \varphi[s]$
- $\mathfrak{M} \models \varphi \rightarrow \psi[s]$  if either  $\mathfrak{M} \not\models \varphi[s]$  or  $\mathfrak{M} \models \psi[s]$

...

- $\mathfrak{M} \models \forall x\varphi[s]$  if for every  $d \in M$ ,  $\mathfrak{M} \models \varphi[s_d^x]$
- $\mathfrak{M} \models \exists x\varphi[s]$  if there is some  $d \in M$ ,  $\mathfrak{M} \models \varphi[s_d^x]$

# Semantics

Note: For  $\varphi(x_1, \dots, x_k)$  (only  $x_1, \dots, x_k$  occur freely) and only the assignments on the exhibited variables matters, therefore we will write say  $\mathfrak{M} \models \varphi(s(x_1), \dots, s(x_k))$  instead of  $\mathfrak{M} \models \varphi[s]$

And the assignment does not even matter at all if  $\sigma$  is a sentence (no free variable). In this case ,we write  $\mathfrak{M} \models \sigma$ , and say  $\sigma$  is **true** in  $\mathfrak{M}$



# Semantics

We are not satisfied with something like “**truth in a structure**”,

~~we~~ we?  
awe

# Semantics

Logical truth

Fix a language  $\mathcal{L}$ , we say a formula / sentence is **valid** (or **logically true**, written  $\models \varphi$ ) if it is true in every  $\mathcal{L}$ -structure

How to decide if a sentence is logically true?

by proof

# Proofs

A proof is a **sequence of formulas**, and every formula in the sequence is an **axiom**, or hypothesized, or derived from some former formulas via some **rules**

# Proofs

## Axioms

(A1) Tautologies (axioms about connectives): e.g.

- $\alpha \rightarrow \beta \rightarrow \alpha$
- $(\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$
- $(\neg\beta \rightarrow \neg\alpha) \rightarrow (\neg\beta \rightarrow \alpha) \rightarrow \beta$

EXE: provide a definition of **tautology**

# Proofs

## Axioms

### (A2) Axioms concerning quantifiers

- $\forall x\alpha \rightarrow \alpha_t^x$ , where variables in  $t$  should not be bounded by quantifier in  $\alpha$  unexpectedly
- $\forall x(\alpha \rightarrow \beta) \rightarrow (\forall x\alpha \rightarrow \forall x\beta)$
- $\alpha \rightarrow \forall x\alpha$ , where  $x$  does not occur freely in  $\alpha$

# Proofs

## Axioms

### (A3) Axioms concerning equality

- $x = x$
- $x = y \rightarrow \alpha \rightarrow \alpha'$ , where  $\alpha'$  is obtained from  $\alpha$  by replacing some free occurrence of  $x$  by  $y$



# Proofs

Fix language, let  $\Sigma$  be a set of formulas and  $\varphi$  a formula. we say  $\varphi$  is **provable** from  $\Sigma$  (or  $\Sigma \vdash \varphi$ ) if **there exists** a sequence of formula  $\langle \beta_0, \dots, \beta_n \rangle$  such that for each  $0 \leq i \leq n$

- $\beta_i$  is among (A1) - (A3) (an logic axiom), or
- $\beta_i \in \Sigma$  (is hypothesised), or
- there exists  $j, k < i$  such that  $\beta_k = \beta_j \rightarrow \beta_i$  ( $\beta_i$  is derived from  $\beta_k$  and  $\beta_j$  via **modus ponens**)  $\beta_n = \varphi$

# Proofs

We write  $\vdash \varphi$  instead of  $\emptyset \vdash \varphi$

Intuitively, provable from nothing meaning logically true

Theorem (Gödel's completeness theorem)

$$\models \varphi \leftrightarrow \vdash \varphi$$

Therefore, the two characterizations of **logical truth** coincide

But logical truth is not the whole story of **mathematical truth**

Even

- $1 + 1 = 2$

- $x \notin x$

are not logically true

Even Even

We want not just to say **arithmetic truth**, **analysis truth**, or **geometry truth** as if there are many different mathematical truths, we want to speak of **mathematical truth**!

A uniform foundation of mathematics is called

# Set theory

$$0 = (0, 0) \quad , \quad -1 = (0, 1) \sim (1, 2)$$
$$1 = (1, 0)$$

Set theory is presumed to be able to talk about all mathematical objects, whatever they are

- natural numbers  $0 = \emptyset$   
 $1 = \{\emptyset\} = \{0\}$
- real numbers  $2 = \{0, 1\}$  --
- relations
- functions

etc. ...

# Set theory

We said  $\mathfrak{N}$  is the standard model for arithmetic. So there is a “target” theory of numbers:  $\text{Th } \mathfrak{N} = \{ \sigma \in \mathcal{L}_A \mid \mathfrak{N} \models \sigma \}$

But what is the **standard model for set theory**?

Set theory is presumed to be about **all sets** and all mathematical objects

But, is the **collection of all sets** a set?



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# Zermelo-Fraenkel Set theory

A formalized set theory is a theory in the

# Zermelo-Fraenkel Set theory

Axiom of extensionality

$$\forall X \forall Y \left( \forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y \right)$$

Axiom of foundation

$$\exists \omega \mid \omega \in X$$

$$\forall X \left( \underline{X \neq \emptyset} \rightarrow \exists y (y \in X \wedge \forall z (z \in X \rightarrow z \notin y)) \right)$$

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Axiom of foundation

$$\forall X (X \neq \emptyset \rightarrow \exists y (y \in X \wedge X \cap y = \emptyset))$$

# Zermelo-Fraenkel Set Theory

Axiom of pair

$$\forall x \forall y \exists Z Z = \{x, y\}$$

where  $z \in \{x, y\}$  stands for  $z = x \vee z = y$

# Zermelo-Fraenkel Set Theory

Axiom of union

$$\forall X \exists Y y = \bigcup X$$

where  $z \in \bigcup X$  is abbr. for  $\exists Y (Y \in X \wedge z \in Y)$

# Zermelo-Fraenkel Set Theory

Axiom of power set

$$\forall X \exists Y \forall Z (Z \in Y \leftrightarrow Z \subset X)$$

Let  $Z \in P(X)$  be an abbr. for  $Z \subset X$ , and  $Y = P(X)$  for  
 $\forall Z (Z \in Y \leftrightarrow Z \subset X)$

# Zermelo-Fraenkel Set Theory

Axiom of power set

$$\forall z(z \subset Z \rightarrow z \subset X)$$
$$\forall X \exists Y \forall Z (Z \in Y \leftrightarrow \underline{Z \subset X})$$

Let  $Z \in P(X)$  be an abbr. for  $Z \subset X$ , and  $Y = P(X)$  for

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# Zermelo-Fraenkel Set Theory

Separation schema: For each formula of the language of set theory  $\varphi(x, v_1, \dots, v_k)$ , the following is an **axiom of separation**

$$\forall A \exists B \forall x (x \in B \leftrightarrow x \in A \wedge \varphi(x, v_1, \dots, v_k))$$

Let  $B = \{x \in A \mid \varphi(x)\}$  be abbr. for  $\forall x (x \in B \leftrightarrow x \in A \wedge \varphi(x))$

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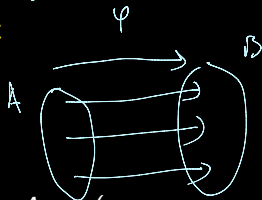
Let  $B = \{x \in A \mid \varphi(x)\}$  be abbr. for  $\forall x(x \in B \leftrightarrow x \in A \wedge \varphi(x))$

# Zermelo-Fraenkel Set Theory

Replacement Schema: For each formula  $\varphi(x, y, v_1, \dots, v_k)$ , the following is an **axiom of replacement**

$$\forall x \exists! y \varphi(x, y, v_1, \dots, v_k)$$

$$\rightarrow \forall A \exists B \forall y (y \in B \leftrightarrow \exists x (x \in A \wedge \varphi(x, y, v_1, \dots, v_k)))$$



# Zermelo-Fraenkel Set Theory

$$\begin{array}{ccccccc} \{ & \{ & \{ & \{ & \{ & \{ & \{ \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \} & \} & \} & \} & \} & \} & \} \end{array} \quad \dots$$

0          1          2          3

Definition

When we say  $X$  is inductive, we mean

$$\emptyset \in X \wedge \forall y (y \in X \rightarrow Sy \in X)$$

Axiom of infinity

$\exists X$   $X$  is inductive

# Zermelo-Fraenkel Set Theory

## Definition

When we say  $X$  is inductive, we mean

$$\emptyset \in X \wedge \forall y(y \in X \rightarrow Sy \in X)$$

Axiom of infinity

$$\exists X X \text{ is inductive}$$

# Zermelo-Fraenkel Set Theory

Axiom of choice

$$\forall A \left( \forall X (X \in A \rightarrow X \neq \emptyset) \rightarrow \forall X \forall Y (X \in A \wedge Y \in A \wedge X \neq Y \rightarrow X \cap Y = \emptyset) \right. \\ \left. \rightarrow \exists B \forall X (X \in A \rightarrow \exists! z z \in B \cap X) \right)$$

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# Zermelo-Fraenkel Set Theory

Axiom of choice

$\forall A$  (elements of  $A$  are nonempty  $\rightarrow A$  is pairwise disjoint  
 $\rightarrow \exists B$   $B$  is a choice set of  $A$ )

# Zermelo-Fraenkel Set Theory

Let **ZFC** be ZF together with the Axiom of choice (**AC**)

# On the horizon of math-truth

But still, this is not enough

Theorem (Gödel's incompleteness theorem)

Assume ZF is consistent. Then there is a sentence  $\sigma$  in the language of set theory such that  $ZF \not\vdash \sigma$  and  $ZF \not\vdash \neg\sigma$ , namely ZF is not complete. Moreover, for every recursive extension  $T \supseteq ZF$ , if  $T$  is consistent, then  $T$  is not complete.

# On the horizon of math-truth

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# On the horizon of math-truth

Theorem (Gödel's 2nd incompleteness theorem)

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# On the horizon of math-truth

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# On the horizon of math-truth

Maybe, every **meaningful** mathematical problem is decidable

# On the horizon of math-truth

## Continuum Hypothesis (CH)

There is no set  $A$  such that

$$\mathbb{N} < A < \mathbb{R}$$

Cantor's Continuum Problem (1st Hilbert Problem):

Is CH true or false?

# On the horizon of math-truth

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# On the horizon of math-truth

## Theorem (Gödel-Cohen)

If ZFC is consistent, then  $ZFC \not\vdash CH$  and  $ZFC \not\vdash \neg CH$

# Next on Set Theory

- Models of theory
- Absoluteness