Set Theory

集合论

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Previously on Set Theory

Cofinality

$[T]$ $p[T]$

- Set theoretical reals:

  Baire Space ω^ω, subtree of $\omega^{<\omega}$

- Complexity of definable sets of reals:

  - Borel sets

    $S^{<}$

    Analytic set Σ^1_1

  - Projective sets

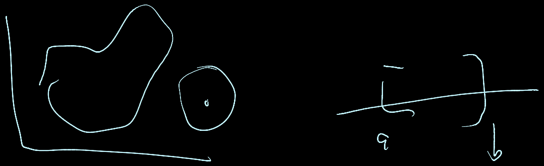
Regularity Properties

Perfect set property

Definition

- A set $P \subset \omega^\omega$ is perfect if it is closed and has no isolated points, i.e. for each (basic) open set $U \subset \omega^\omega$ and $x \in P$, if $x \in U$, then $U \cap (P \setminus \{x\}) \neq \emptyset$
- A set $A \subset \omega^\omega$ has the perfect set property if it is either countable or contains a nonempty perfect subset

Perfect set property



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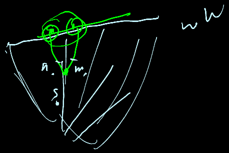
Perfect set property

Embed cantor space into a perfect set

2^ω

Fact

If P is perfect, then card $P = 2^{\aleph_0}$. Thus, if $A \subset \omega^\omega$ has the perfect set property, then A cannot be a witness of the failure of CH



$f\langle 0\rangle = s_0{}^\frown n_0$

$f\langle 1\rangle = s_0{}^\frown m_0$

$f\langle 0,0\rangle = s_{00}{}^\frown n_{00}$ $f\langle 1,0\rangle = s_{01}{}^\frown n_{01}$

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Perfect set property

Theorem (Cantor-Bendixson)

Every closed set has the profect set property

Let $C \subset \omega^\omega$ be closed.

Show that C is either countable or contains a nonempty perfect set

Proof Given C_0, Let $C'_\alpha = \{ a \in C_\alpha \mid a \text{ is an isolated point in } C_\alpha \}$

Let $C_1 = C_0 - C'_0$

$C_{\alpha+1} = C_\alpha - C'_\alpha$

$C_\alpha = \bigcap_{\xi < \alpha} C_\xi$ for limit α

Claim $\exists \alpha$ $C_\alpha = C_{\alpha+1}$

Trivial

Let $P = C_\alpha$ where α is the least s.t. $C_\alpha = C_{\alpha+1}$

clearly P is perfect

Claim $P - C$ is at most countable

Let $\{ I_k : k < \omega \}$ enumerate all open interval with rational ends

For each $x \in P - C$, there is some $\xi < \alpha$ $x \in C'_\xi$, an isolated point in C_ξ

so there is the least $k(x) \in \omega$ s.t. $I_{k(x)} \cap C_\xi = \{x\}$

subclaim $k : P - C \to \omega$ is 1-1

clearly $C'_\xi \cap C'_\theta = \emptyset$ if $\xi \neq \theta < \alpha$

if $x, y \in C'_\xi$, then $k(x) = k(y)$

if $x \in C'_\xi$, $y \in C'_\theta$ where $\xi < \theta$, then $y \in C_\xi$, so $y \notin I_{k(x)}$, but $y \in I_{k(y)}$

so $k(x) \neq k(y)$ □

Perfect set property

Fact

Every analytic set has perfect set property

Perfect set property

Definition

A set $B \subset \omega^\omega$ is Bernstein if neither B nor $\omega^\omega \setminus B$ contains a perfect subset

Fact (AC)

There is a Bernstein set, which violates the perfect set property

Construct a Bernstein set

card $\{C \mid C \text{ is closed}\} = \text{card}\{[T] \mid T \subseteq \omega^{<\omega}\}$

There are at most $2^{\aleph_0}$ perfect set

Let $\{P_\xi : \xi < 2^{\aleph_0}\}$ be the enumeration of all nonempty perfect set (AC)

Let (a_0, b_0) be $a_0 \neq b_0$, $a_0, b_0 \in P_0$

Let $A_0 = \{a_0\}$, $B_0 = \{b_0\}$

In general

Let $P^*_\xi = P_\xi - \bigcup_{\theta<\xi} A_\theta - \bigcup_{\theta<\xi} B_\theta$,

By induction. card $P^*_\xi = 2^{\aleph_0} - \text{card}\,\xi = 2^{\aleph_0}$ $[\xi < 2^{\aleph_0}]$

Let a_ξ, b_ξ be s.t. $a_\xi \neq b_\xi$ and $a_\xi, b_\xi \in P^*_\xi$

And $A_\xi = \bigcup_{\theta<\xi} A_\theta \cup \{a_\xi\}$, $B_\xi = \bigcup_{\theta<\xi} B_\theta \cup \{b_\xi\}$

Let $A = \bigcup_{\xi<2^{\aleph_0}} A_\xi$

$B = \bigcup_{\xi<2^{\aleph_0}} B_\xi$

Then A, B are Bernstein set

Perfect set property

Fact

- If $V = L$, then there is a $\mathbf{\Pi}^1_1$ set of reals which does not have the perfect set property
- It follows from a large cardinal axiom that every projective set of reals has the perfect set property

Lesbesgue Measurable

For now, We consider the real line $\mathbb{R}$

Definition

Outer Measure

- For interval $I = (a, b)$, define the length of the interval$l(I) = b - a$
- For $A \subset \mathbb{R}$, the outer measure of A is

$$\mu^*(A) = \inf\left\{ \sum_{k\in\omega} l(I_k) \,\middle|\, \langle I_k \rangle_{k\in\omega} \text{ is a open covering of } A \right\}$$

Lesbesgue Measurable

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Definition

Outer Measure

- For interval $I = (a, b)$, define the length of the interval $l(I) = b - a$
- For $A \subset \mathbb{R}$, the outer measure of A is

$A \subseteq \bigcup_{k \in \omega} I_k$

$$\mu^*(A) = \inf\left\{ \sum_{k \in \omega} l(I_k) \;\middle|\; \langle I_k \rangle_{k \in \omega} \text{ is a open covering of } A \right\}$$

Lesbesgue Measurable

Definition

Lesbesgue Measure

- A set $A \subset \mathbb{R}$ is (Lesbesgue) measurable if for every $X \subset \mathbb{R}$

$$\mu^*(A) = \mu^*(A \cap X) + \mu^*(A \setminus X)$$

- For Lesbesgue measurable set A, its Lesbesgue measure $\mu(A) = \mu^*(A)$

Lesbesgue Measurable

Fact

The collection of Lesbesgue measurable sets forms a σ-algebra. Thus every Borel set is Lesbesgue measurable

Fact Lesbesgue measure is σ-additive and translation inverant

Theorem (Vitali)

There is a nonmeasurable set of reals

We find a Vitali set which is not measurable

For $x, y \in (0,1)$

$x \sim y$ iff $x - y \in \mathbb{Q}$

Consider $\{ [x]_\sim \mid x \in (0,1) \} = \mathcal{F}$

Each $[v]$ is countable

so there are $2^{\aleph_0}$ many equivalent classes

By AC, There is a choice set V

of $\mathcal{F}$. i.e. $V \cap [x]_\sim$ is singleton for $x \in (0,1)$

so card $V = 2^{\aleph_0}$

We say V is a Vitali set

Claim V is not measurable

$$\bigcup_{r \in (-1,1) \cap \mathbb{Q}} V + r \supseteq (0,1), \quad \subseteq (-1, 2)$$

$$1 \leq \mu\Big(\bigcup_{r \in (-1,1) \cap \mathbb{Q}} V + r \Big) \leq 3$$

μ cannot be σ-additive and translation invariant on V

Lesbesgue Measurable

Fact

Assuming $V = L$, there is a $\mathbf{\Sigma}^1_2$ set of reals which is not measurable

Fact

It is provable from a large cardinal axiom that every projective set is measuable

Property of Baire

Definition

- A set $X \subset \omega^\omega$ is **nowhere dense** if its closure has empty interior, i.e. no nonempty open set is contained in its closure
- A set $M \subset \omega^\omega$ is **meager** if its is the union of countably many nowhere dense sets
- A set $A \subset \omega^\omega$ is said to have the **property of Baire** if it differs from an open set by a meager set

Almost open

Property of Baire

Fact

The family of sets with the property of Baire forms a σ-algebra, which follows that every Borel set has the property of Baire

Fact

Neither Bernstein set nor Vitali set has the property of Baire

Property of Baire

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Player I a_0 a_1

Player II b_0 b_1 ...

Determinacy

Definition (The game on ω)

Let $A \subset \omega^\omega$. We define the game G_A as follow

- A play is a sequence $\langle a_0, b_0, a_1, b_1, \ldots \rangle$ $\in \omega^\omega$ of natural numbers, where $\langle a_0, a_1, \ldots \rangle$ is the play of player I and $\langle b_0, b_1, \ldots \rangle$ is the play of player II
- Player I wins the play if $\langle a_0, b_0, a_1, b_1, \ldots \rangle \in A$

Determinacy

Definition (The game on ω)

- A strategy is a function $\sigma : \omega^{<\omega} \to \omega$. Given $b = \langle b_0, b_1, \ldots \rangle$, $\sigma * b = \langle \sigma(\emptyset), b_0, \sigma(\langle \sigma(\emptyset), b_0 \rangle), b_1, \ldots \rangle$ is the play of the player I's strategy σ against b. Similar is defined for player II's $a * \sigma$
- A winning strategy for player I is a strategy σ such that $\sigma * b \in A$ for all $b \in \omega^\omega$
- A winning strategy for player II is a strategy σ such that $a * \sigma \notin A$ for all $a \in \omega^\omega$

Determinacy

Definition

Axiom of determinacy

- A set $A \subset \omega^\omega$ is determined if either player I or player II has a winning strategy in G_A
- Axiom of determinacy (AD): Every subset $A \subset \omega^\omega$ is determined

Determinacy

Fact

$\mathbf{\Pi}^1_n$-determinacy implies that every $\mathbf{\Sigma}^1_{n+1}$ set is Lebesgue measurable, has the property of Baire and the perfect set property

Corollary (AC)

$\neg$AD

Determinacy

Fact

$\mathbf{\Pi}^1_n$-determinacy implies that every $\mathbf{\Sigma}^1_{n+1}$ set is Lebesgue measurable, has the property of Baire and the perfect set property

~~Corollary~~ Fact (AC)

$\neg$AD

Determinacy

Fact

$\mathbf{\Pi}^1_n$-determinacy implies that every $\mathbf{\Sigma}^1_{n+1}$ set is Lebesgue measurable, has the property of Baire and the perfect set property

Corollary

If every projective set is determined (PD), then every projective set has the regularity properties

Determinacy

Theorem

Martin-Steel The existence of n Woodin cardinal with a measurable cardinal above them all imply $\mathbf{\Pi}^1_{n+1}$-determinacy. Therefore the existence of infinitely many Woodin cardinals implies PD

Next on Set Theory

- Constructible sets (Gödel's L)
- Forcing
- More on large cardinals and descriptive set theory