

Set Theory

集合论

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Previously on Set Theory

Cofinality

\aleph_1

\aleph_1

- Set theoretical reals:

Baire Space ω^ω , subtree of $\omega^{<\omega}$

- Complexity of definable sets of reals:

- Borel sets

\aleph_1

- Projective sets

Analytic set Σ_1^1

Regularity Properties

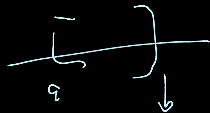
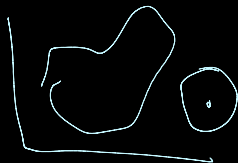
Perfect set property

Definition

- A set $P \subset \omega^\omega$ is **perfect** if it is closed and has no isolated points, i.e. for each (basic) open set $U \subset \omega^\omega$ and $x \in P$, if $x \in U$, then $U \cap (P \setminus \{x\}) \neq \emptyset$
- A set $A \subset \omega^\omega$ has the perfect set property if it is either countable or contains a nonempty perfect subset

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- A set $A \subset \omega^\omega$ has the **perfect set property** if it is either countable or contains a nonempty perfect subset

Perfect set property

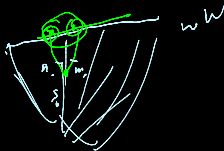
Embed cantor space into a perfect set

$$2^\omega$$

Fact

$$\omega \vee 1$$

If P is perfect, then $\text{card } P = 2^{\aleph_0}$. Thus, if $A \subset \omega^\omega$ has the perfect set property, then A cannot be a witness of the failure of CH



$$f \langle 0 \rangle = s_0 \hat{\ } n_0$$

$$f \langle 1 \rangle = s_0 \hat{\ } m_0$$

$$f \langle 0, 0 \rangle = s_{00} \hat{\ } n_{00} \quad f \langle 1, 0 \rangle = s_{01} \hat{\ } n_{01}$$

$$f \langle 0, 1 \rangle = s_{00} \hat{\ } m_{00} \quad f \langle 1, 1 \rangle = s_{01} \hat{\ } m_{01}$$

Perfect set property

Theorem (Cantor-Bendixson)

Every closed set has the perfect set property

Let $C \subset \omega^\omega$ be closed.

Show that C is either countable or contains a nonempty perfect set

Proof Given C_0 , let $C'_0 = \{a \in C \mid a \text{ is an isolated point}\}$

$$\text{Let } C_1 = C_0 - C'_0$$

$$C_{n+1} = C_n - C'_n$$

$$C_\omega = \bigcap_{1 \leq n} C_n$$

Claim $\exists n$ $C_n = C_{n+1}$

Assume $C_\omega \neq \emptyset$,

Note: each C'_n is countable

(each isolated point is

witnessed by a (r, q) , where
 $r, q \in \mathbb{Q}$]

Perfect set property

Fact

Every analytic set has perfect set property

Perfect set property

Definition

A set $B \subset \omega^\omega$ is **Bernstein** if neither B nor $\omega^\omega \setminus B$ contains a perfect subset

Fact (AC)

There is a Bernstein set, which violates the perfect set property

Construct a Bernstein set

$$[7] \quad \mathbb{T} \subseteq \omega^{\omega}$$

There are at most 2^{\aleph_0} perfect set

Let $\{P_s : s \in \mathbb{N}\}$ be the enumeration of all ^{nonempty} perfect set ($A \subset \mathbb{T}$)

Let (a_s, b_s) be $a_s \neq b_s, a_s, b_s \in P_s$

Let $A_s = \{a_s\}, B_s = \{b_s\}$

Let $P_{s+1}^* = P_s^* - A_s - B_s$

Then $\text{card } P_{s+1}^* = 2^{\aleph_0} - \text{card } \{a_s, b_s\} = 2^{\aleph_0}$ ($s < \mathbb{N}$)

Let a_s, b_s be s.t. $a_s \neq b_s$ and $a_s, b_s \in P_s^*$

And $A_{s+1} = A_s \cup \{a_s\}, B_{s+1} = B_s \cup \{b_s\}$

$$\text{card } \{ \mathbb{T} \subseteq \omega^{\omega} \} =$$

$$\text{card } \{ X \subseteq \omega \}$$

$$\text{Let } A = \bigcup_{s \in \mathbb{N}} A_s$$

$$B = \bigcup_{s \in \mathbb{N}} B_s$$

Then A, B are Bernstein set

limit cover are ω -initially defined

Perfect set property

Fact

- If $V = L$, then there is a \aleph_1^1 set of reals which does not have the perfect set property
- It follows from a large cardinal axiom that every projective set of reals has the perfect set property

Lesbesgue Measurable

For now, We consider the real line \mathbb{R}

Definition

Outer Measure

- For interval $I = (a, b)$, define the length of the interval $l(I) = b - a$
- For $A \subset \mathbb{R}$, the outer measure of A is

$$\mu^*(A) = \inf \left\{ \sum_{k \in \omega} l(I_k) \mid \langle I_k \rangle_{k \in \omega} \text{ is a open covering of } A \right\}$$

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$A \subseteq \bigcup_{k \in \omega} I_k$

Lesbesgue Measurable

Definition

Lesbesgue Measure

- A set $A \subset \mathbb{R}$ is (Lesbesgue) measurable if for every $X \subset \mathbb{R}$

$$\mu^*(A) = \mu^*(A \cap X) + \mu^*(A \setminus X)$$

- For Lesbesgue measurable set A , its Lesbesgue measure $\mu(A) = \mu^*(A)$

Lesbesgue Measurable

Fact

The collection of Lesbesgue measurable sets forms a σ -algebra. Thus every Borel set is Lesbesgue measurable

Theorem (Vitali)

There is a nonmeasurable set of reals

We find a Vitali set which is not measurable

\mathcal{T}

Lesbesgue Measurable

Fact

Assuming $V = L$, there is a Σ_2^1 set of reals which is not measurable

Fact

It is provable from a large cardinal axiom that every projective set is measurable

Property of Baire

Definition

- A set $X \subset \omega^\omega$ is **nowhere dense** if its closure has empty interior, i.e. no nonempty open set is contained in its closure
- A set $M \subset \omega^\omega$ is **meager** if it is the union of countably many nowhere dense sets
- A set $A \subset \omega^\omega$ is said to have the **property of Baire** if it differs from an open set by a meager set

Property of Baire

Fact

The family of sets with the property of Baire forms a σ -algebra, which follows that every Borel set has the property of Baire

Fact

Neither Bernstein set nor Vitali set has the property of Baire

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Determinacy

Definition (The game on ω)

Let $A \subset \omega^\omega$. We define the game G_A as follow

- A **play** is a sequence $\langle a_0, b_0, a_1, b_1, \dots \rangle$ of natural numbers, where $\langle a_0, a_1, \dots \rangle$ is the **play of player I** and $\langle b_0, b_1, \dots \rangle$ is the **play of player II**
- **Player I wins** the play if $\langle a_0, b_0, a_1, b_1, \dots \rangle \in A$

Determinacy

Definition (The game on ω)

- A **strategy** is a function $\sigma : \omega^{<\omega} \rightarrow \omega$. Given $b = \langle b_0, b_1, \dots \rangle$, $\sigma * b = \langle \sigma(\emptyset), b_0, \sigma(\langle \sigma(\emptyset), b_0 \rangle), b_1, \dots \rangle$ is the play of the player I's strategy σ against b . Similar is defined for player II's $a * \sigma$
- A **winning strategy for player I** is a strategy σ such that $\sigma * b \in A$ for all $b \in \omega^\omega$
- A **winning strategy for player II** is a strategy σ such that $a * \sigma \notin A$ for all $a \in \omega^\omega$

Determinacy

Definition

Axiom of determinacy

- A set $A \subset \omega^\omega$ is **determined** if either player I or player II has a winning strategy in G_A
- Axiom of determinacy (AD): Every subset $A \subset \omega^\omega$ is determined

Determinacy

Fact

Π_n^1 -determinacy implies that every Σ_{n+1}^1 set is Lebesgue measurable, has the property of Baire and the perfect set property

Corollary (AC)

\neg AD

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Corollary

If every projective set is determined (**PD**), then every projective set has the regularity properties

Determinacy

Theorem

Martin-Steel The existence of n Woodin cardinal with a measurable cardinal above them all imply Π_{n+1}^1 -determinacy. Therefore the existence of infinitely many Woodin cardinals implies PD

Next on Set Theory

- Constructible sets (Gödel's L)
- Forcing
- More on large cardinals and descriptive set theory