Set Theory

集合论

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Previously on Set Theory

Cofinality

- A weakly compact cardinal is Mahlo, and the set of Mahlo cardinals below is stationary
- Ultraproduct and compactness theorem
- A measurable cardinal is weakly compact

Set theoretical real numbers

Definition (Topology space)

A topology space is an ordered pair (X, Γ), such that $\Gamma \subset P(X)$ satisfies

- $\emptyset, X \in \Gamma$
- For each $F \subset \Gamma$, $\bigcup F \in \Gamma$
- If $U, V \in \Gamma$, then $U \cap V \in \Gamma$

We say Γ is a topology on X and sets in Γ are open sets. The complements of open sets are closed sets.

Set theoretical real numbers

Definition (Ordinary (order) topology on real numbers)

The ordinary topology on $\mathbb{R}$ is generated from the collection of open intervals. I.e. A set $U \subset \mathbb{R}$ is open if and only if for each $r \in U$, there is $x, y \in \mathbb{R} \cup \{\infty, -\infty\}$ such that $r \in (x, y) \subset U$. We also say the set

$$\left\{(x, y) \,\middle|\, x, y \in \mathbb{R} \cup \{\infty, -\infty\}\right\}$$

is a basis for the normal topology on $\mathbb{R}$

Set theoretical real numbers

Fact

There is a countable basis for the normal topology on $\mathbb{R}$

EXE:

Set theoretical real numbers

Definition (Baire space)

- Fix set $X \neq \emptyset$. For $s \in X^{<\omega}$, define $s^{\prec} = \left\{x \in X^{\omega} \mid s \subset x\right\}$
- We define the ordinary topology on X^{ω} as generated from the basis

$$\left\{s^{\prec} \mid s \in X^{<\omega}\right\}$$

- Baire space is defined to be the ordinary topology on ω^{ω}

Set theoretical real numbers

Fact

- A basic open set $s^{\prec}$ is also a closed set.
- $U \subset \omega^{\omega}$ is open if and only if it is the (countable) union of a collection of basic open sets $\bigcup_{n \in \omega} s_n^{\prec}$

Set theoretical real numbers

Definition (Cantor space)

Cantor space is the ordinary topology space on 2^ω

Set theoretical real numbers

Descriptive set theorists who are interested in the complexity of definable sets of reals usually works on Baire space or Cantor space rather than the ordinary space on the real line. Actually, Baire space, Cantor space and real line are all Polish spaces, i.e. separable completely metrizable topology space.

Fact

Baire space is homeomorphic to the order space of all irrational numbers

EXE*: [Hint: use continued fraction]

Set theoretical real numbers

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Fact

Baire space is homeomorphic to the order space of all irrational numbers

EXE*: [Hint: use continued fraction]

Set theoretical real numbers

Definition

- A tree T on X is a subset of $X^{<\omega}$ ordered by $\subset$ and closed under initial segments.
- Let T be a tree on X. We define
  $[T] = \left\{x \in X^\omega \,\middle|\, \forall n\ x\restriction n \in T\right\}$

Set theoretical real numbers

$\Leftarrow$

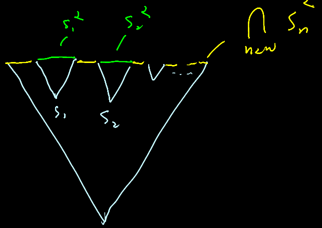
A is closed if $A = \overline{\bigcup_{n\in\omega} s_n^2} = \bigcap_{n\in\omega} \overline{s_n^2}$

Let $T = \{ t \in \omega^{<\omega} \mid \forall n\ t \not\supset s_n \}$

Then $x \in [T] \Leftrightarrow \forall m\ x\restriction m \in T$

$\Leftrightarrow \forall n \forall m\ x\restriction m \not\supset s_n$

$\Leftrightarrow \forall n\ x \not\supset s_n \Leftrightarrow x \in \bigcap_n \overline{s_n^2}$



Fact

$A \subset \omega^\omega$ is closed if and only there is a tree T on ω such that

$A = [T]$

($\Rightarrow$) trivial

Borel sets

Definition

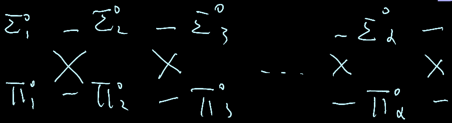
- A σ-algebra on a set Y is a collection $S \subset P(Y)$ which is closed under complement, countable union and countable intersection
- The collection of Borel set are defined to be the smallest σ-algebra containing all open sets of the Biare space

Borel sets

Definition (Borel hierarchy)

- $\mathbf{\Sigma}^0_1$ = the collection of open sets
- $\mathbf{\Pi}^0_1$ = the collection of closed sets
- $\mathbf{\Sigma}^0_\alpha$ = the collection of all sets $A = \bigcup_{n\in\omega} A_n$, where each A_n is $\mathbf{\Pi}^0_\beta$ set for some $\beta < \alpha$
- $\mathbf{\Pi}^0_\alpha$ = the collection of all complements of $\mathbf{\Sigma}^0_\alpha$ sets

Borel sets



Fact

- $\mathbf{\Pi}^0_\alpha$ = the collection of all sets $A = \bigcap_{n\in\omega} A_n$, where each A_n is $\mathbf{\Sigma}^0_\beta$ set for some $\beta < \alpha$
- If $\alpha < \beta$, then $\mathbf{\Sigma}^0_\alpha \subset \mathbf{\Sigma}^0_\beta$, $\mathbf{\Sigma}^0_\alpha \subset \mathbf{\Pi}^0_\beta$, $\mathbf{\Pi}^0_\alpha \subset \mathbf{\Pi}^0_\beta$, and $\mathbf{\Pi}^0_\alpha \subset \mathbf{\Sigma}^0_\beta$
- $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$ is the collection of Borel sets
- EXE*: $\mathbf{\Sigma}^0_\alpha \not\subset \mathbf{\Pi}^0_\alpha$ $(1 \leq \alpha < \omega_1)$

Projective sets

Definition

The ordinary topology on $(\omega^\omega)^k$ is defined to be the product topology.

Fact

$(\omega^\omega)^k$ is homeomorphic to ω^ω

Projective sets

$\pi : \omega \times \omega \to \omega$

Definition

The ordinary topology on $(\omega^\omega)^k$ is defined to be the product topology.

$\omega^\omega \times \omega^\omega \to \omega^\omega$

$(f, g) \to h$

Fact

$(\omega^\omega)^k$ is homeomorphic to ω^ω

$h(n) = \pi(f(n), g(n))$

Projective sets

Definition

- T is a tree on $X \times Y$, if T is the subset of

  $\left\{(s,t) \,\middle|\, s \in X^{<\omega}, t \in Y^{<\omega}, |s| = |t|\right\}$

  Let T be a tree on $X \times Y$

- Define $[T] = \left\{(x,y) \in X^{\omega} \times Y^{\omega} \,\middle|\, \forall n \; (x \restriction n, y \restriction n) \in T\right\}$

- Define the projection of T,

  $p[T] = \left\{x \in X^{\omega} \,\middle|\, \exists y (x,y) \in [T]\right\}$

  $\exists y \; \forall n \; (x \restriction n, y \restriction n) \in T$

Projective sets

Definition (Analytic set)

A set $A \subset \omega^\omega$ is called analytic if there is a tree T on $\omega \times \omega$ with $A = p[T]$. coanalytic sets are complements of analytic sets

Lemma

Every Borel set is an analytic set

Proof.

It is sufficient to show

- Every basic open set is analytic
- Every closed set is analytic
- Analytic sets are closed under countable union
- Analytic sets are closed under countable intersection

Projective sets

Lemma

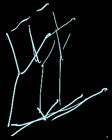
$S^i = p[T]$ where

$T = \{(s', t') \mid s' | s, |s| = |t|\}$

Every Borel set is an analytic set

Proof.

It is sufficient to show



- Every basic open set is analytic
- Every closed set is analytic ($\bigcap_{n \in \omega} S^i_n$)

  $T = \{(s', t') \mid \forall n\ s' \not| s_n, |s| = |t|\}$

  $p[T] = \bigcap_{n \in \omega} S^i_n$
- Analytic sets are closed under countable union
- Analytic sets are closed under countable intersection

Analytic sets are closed under countable union

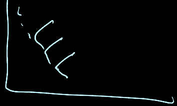
$A = \bigcup_{n \in \omega} A_n$, where $A_n = p[T_n]$

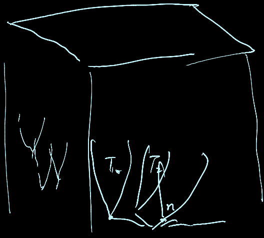
Find T s.t. $A = p[T]$

If $(t, s) \in T_n$, we put $(t, \langle n \rangle ^\frown s_0)$ into T where $s_0 = s \upharpoonright |s|-1$, i.e.

$T = \{ (t, \langle n \rangle ^\frown s_0) \mid \exists s, (t,s) \in T_n$ and $s_0 = s \upharpoonright |s|-1 \}$

Note: $\bigcup_n T_n$ disjoint trick





If $f \in p[T_n]$ for some n, then $\exists g$ s.t. $\forall m \ (f \upharpoonright m, g \upharpoonright m) \in T_n$

Let $g^* = \langle n \rangle ^\frown g$, we have $\forall m, (f \upharpoonright m, g^* \upharpoonright m) \in T$, where $g^* \upharpoonright m = \langle n \rangle ^\frown g \upharpoonright m-1$

Analytic sets are closed under countable intersection

Let $A = \bigcap_{n \in \omega} A_n$ s.t. $A_n = p[T_n]$

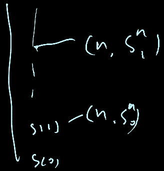
Let $\pi : \omega \times \omega \to \omega$ be the canonical coding

Fix $s \in \omega^k$

Define $s^n = \{ s^n_0, s^n_1 \dots s^n_j \}$ s.t.

$\pi(j, s^n_j) \in \operatorname{ran} s$

$T = \{ (t, s) \mid \forall n \ (t \upharpoonright |s^n|, s^n) \in T_n \}$



Projective sets

Theorem (Souslin)

A set $A \subset \omega^\omega$ is Borel if and only if it is analytic and coanalytic

Projective sets

Theorem

There is an analytic set which is not coanalytic

Lemma

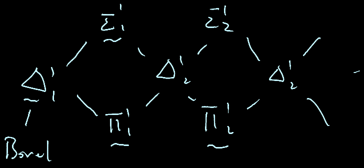
- There is a universal analytic set, i.e. a analytic set $B \subset \omega^\omega \times \omega^\omega$ such that for any analytic set $A \subset \omega^\omega$ there is some $g \in \omega^\omega$ with $A = \left\{ f \middle| (f, g) \in B \right\}$.
- A universal analytic set is not coanalytic

Projective sets

Definition (The hierarchy of projective sets)

- $\mathbf{\Sigma}^1_1$ = the collection of all analytic sets
- $\mathbf{\Pi}^1_n$ = the complements of $\mathbf{\Sigma}^1_n$ sets
- $\mathbf{\Sigma}^1_{n+1}$ = the collection of projections of $\mathbf{\Pi}^1_n$ sets
- $\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$

Projective sets



Fact

- $\mathbf{\Delta}^1_n \subset \mathbf{\Sigma}^1_n \subset \mathbf{\Delta}^1_{n+1} \subset \mathbf{\Sigma}^1_{n+1}$
- $\mathbf{\Delta}^1_n \subset \mathbf{\Pi}^1_n \subset \mathbf{\Delta}^1_{n+1} \subset \mathbf{\Pi}^1_{n+1}$

$\Sigma^1_n \subseteq \Sigma^1_{n+1}$

$\Sigma^1_n \subseteq \Pi^1_{n+1} \Rightarrow \Sigma^1_n \subseteq \Delta^1_{n+1}$

$\Sigma_{n-1} \subseteq \Sigma^1_n$

$\Pi^1_1 \subseteq \Sigma^1_2$

$\Pi^1_n \Rightarrow \Pi^1_{n+1}$

Definition

A set $A \subset \omega^\omega$ is projective if $A \in \Sigma^1_n$ for some $n \in \omega$

Projective sets

Fact

- $\mathbf{\Delta}^1_n \subset \mathbf{\Sigma}^1_n \subset \mathbf{\Delta}^1_{n+1} \subset \mathbf{\Sigma}^1_{n+1}$
- $\mathbf{\Delta}^1_n \subset \mathbf{\Pi}^1_n \subset \mathbf{\Delta}^1_{n+1} \subset \mathbf{\Pi}^1_{n+1}$

Definition

A set $A \subset \omega^\omega$ is projective if $A \in \Sigma^1_n$ for some $n \in \omega$

Next on Set Theory

Regularity properties