Set Theory

集合论

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

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Previously on Set Theory

Large cardinal

- Inaccessible cardinal
- Mahlo cardinal

κ is Mahlo : if κ is inaccessible

$\{\lambda < \kappa \mid \lambda$ is regular$\}$ is stat. in κ

Previously on Set Theory

Tree

- Tree, level, height
- chain, maximal chain, cofinal chain, antichain
- κ-tree, κ-Aronszajn tree, κ-Souslin tree

Previously on Set Theory

Weakly compact cardinal

- There is no ω-Aronszajn tree
- There is a ω_1-Aronszajn tree
- Weakly compact cardinal: κ is inaccessible and there is no κ-Aronszajn tree

Weakly compact cardinal

Definition

A cardinal κ is weakly compact if it is inaccessible and every κ-tree has a cofinal chain

Weakly compact cardinal

Lemma

Let A be a set of uncountable cardinals such that for all regular cardinal λ, $A \cap \lambda$ is not stationary in λ. Then there is a regressive injection $t : A \to \mathrm{OR}$

EXE:

Weakly compact cardinal

Lemma

If κ is weakly compact and S is a stationary subset of κ, then there is a regular $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in λ

proof: If κ is inaccessible then $\{\mu < \kappa \mid \mu \text{ is cardinal}\}$ is club

Assume to contradiction that S is a stationary subset in κ (containing only uncountable cardinals) such that for each regular $\lambda < \kappa$ $S \cap \lambda$ is non-stationary, we build a κ-Aronszajn tree.

Let $T \subseteq \kappa^{<\kappa}$ be s.t. For $t \in \kappa^{\alpha}$

$t \in T$ iff $t \restriction (S \cap \alpha)$ is regressive and 1-1, And for $\xi \in \alpha \setminus S$ $t(\xi) = 0$

① T is tree

② For $\alpha < \kappa$, $|L_{\alpha}^{T}| < \kappa$

[since if $t \in L_{\alpha}^{T} \subseteq \kappa^{\alpha}$, then $t \in \alpha^{\alpha}$, So $|L_{\alpha}^{T}| \leq |2^{\alpha}| < \kappa$]

③ T has no cofinal chain

[For otherwise, let c is a cofinal chain. Consider $f = \bigcup c$. $f \restriction S : S \to \kappa$ is regressive and 1-1. But club filter is normal]

④ height$(T) = \kappa$

By former Lemma

Weakly compact cardinal

Corollary

If κ is a weakly compact cardinal, then κ is Mahlo. Moreover, there is a stationary set of Mahlo cardinals below κ

① $\{\lambda < \kappa \mid \lambda \text{ is inaccessible}\}$ and

② $\{\lambda < \kappa \mid \lambda \text{ is Mahlo}\}$ are stationary

$\leadsto S_2$

Proof

① Consider $\{\lambda < \kappa \mid \lambda \text{ is regular}\} = S_0$

We show S_0 is stationary

Fix a club C. since C is stat.

so there is regular $\lambda < \kappa$, s.t. $C \cap \lambda$

is stat in λ

therefore $\lambda \in C$, and $\lambda \in S_0$

i.e. $C \cap S_0 \neq \emptyset$ so,

κ is Mahlo

② Since κ is Mahlo,

$S_1 = \{\lambda < \kappa \mid \lambda \text{ is inaccessible}\}$ is stat

Fix C, the $C \cap S_1$ is stationary

so, there is regular $\lambda < \kappa$, s.t.

$C \cap S_1 \cap \lambda$ is stat. in λ

Therefore λ is Malho

and $\lambda \in C$,

i.e. $\lambda \in C \cap S_2$

Compactness theorem ultraproduct

Definition

Infinitary logic Let λ, μ be infinite cardinals. An $\mathcal{L}_{\lambda\mu}$ language are defined allowing infinite conjunctions $\bigwedge_{\xi<\alpha}$ $(\alpha < \lambda)$ and infinite quantifier sequence $\forall_{\xi<\beta}$ $(\beta < \mu)$. A formula in a $\mathcal{L}_{\lambda\mu}$ language can have less than μ free variables.

Satisfaction relation are defined intuitively

Compactness theorem ultraproduct

Fact

κ is weakly compact if and only if the infinitary logic $\mathcal{L}_{\kappa\kappa}$ satisfies the weak compactness theorem, i.e. whenever Σ is a set of sentence in $\mathcal{L}_{\kappa\kappa}$ of cardinality at most κ and every subset with cardinality less than κ is satisfiable, then Σ is satisfiable

Definition

κ is strongly compact if $\mathcal{L}_{\kappa\kappa}$ satisfies the compactness theorem, which is without the restriction on the cardinality of Σ.

Compactness theorem ultraproduct

Fact

κ is weakly compact if and only if the infinitary logic $\mathcal{L}_{\kappa\kappa}$ satisfies the weak compactness theorem, i.e. whenever Σ is a set of sentence in $\mathcal{L}_{\kappa}$ of cardinality at most κ and every subset with cardinality less than κ is satisfiable, then Σ is satisfiable

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Compactness theorem ultraproduct

Theorem

$\mathcal{L}_{\omega\omega}$ have compactness theorem

Compactness theorem ultraproduct

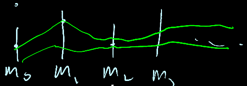
$i \in I$ $F_i = \{B \mid i \in B\}$

$A \subseteq I$ $U \supseteq F_A = \{B \mid B \supseteq A\}$

$|A| < |I|$

Definition (Ultraproduct)

Let I be a index set, $\mathfrak{M} = \left\{M_i \mid i \in I\right\}$ is a I-sequence structures in a given language, and U is an non-principle ultrafilter on I. We define the ultraproduct $\prod_{i \in I} M_i/U$ to be the following structure:



- The universe is $\left\{[f]_U \mid f \in (\bigcup_{i \in I} |M_i|)^I \text{ and } f(i) \in |M_i|\right\}$, where $f \sim_U g$ if the set $\left\{i \in I \mid f(i) = g(i)\right\} \in U$

Compactness theorem ultraproduct

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- For relation symbol, e.g. R: $[f]_U R^{\prod_{i \in I} M_i / U} [g]_U$ if the set $\left\{ i \in I \mid f(i) R^{M_i} g(i) \right\} \in U$

Compactness theorem ultraproduct

Definition (Ultraproduct)

Let I be a index set, $\mathfrak{M} = \left\{ M_i \;\middle|\; i \in I \right\}$ is a I-sequence structures in a given language, and U is an non-principle ultrafilter on I.

We define the ultraproduct $\prod_{i \in I} M_i / U$ to be the following structure:

- For function symbol, say π: Define $\pi^{\prod_{i \in I} M_i / U}\left([f]_U\right) = [g]_U$ where $g(i) = \pi^{M_i}(f(i))$

Compactness theorem ultraproduct

Theorem (Łoś)

Let $M = \prod_{i \in I} M_i / U$ be an ultraproduct and $\varphi(x_1, \ldots, x_n)$ is a formula. Then

$$M \vDash \varphi([f_1], \ldots, [f_n]) \Leftrightarrow \left\{ i \in I \,\middle|\, M_i \vDash \varphi\big(f_1(i), \ldots, f_n(i)\big) \right\} \in U$$

Pmf IND on φ

$M \vDash [f] = [g]$

$M \vDash [f] \; R^M [g]$

$M \vDash \varphi \wedge \psi$ iff $M \vDash \varphi$ & $M \vDash \psi$

$M \vDash \neg\varphi$ iff Not $M \vDash \varphi$

⋮

Compactness theorem: If Σ is finitely satisfiable, then Σ is satisfiable

Let $I = \{ \delta \mid \delta \subseteq \Sigma$ is finite $\}$ → Let $I = \{ \delta_i \mid i \in \omega \}$ s.t. $\delta_i \subseteq \Sigma_i$

We know for each $\delta \in I$, there is a model $M_\delta \models \delta$

Let U be an non-principl ultrafilter on I

[Note. $|I| = |\omega^{<\omega}| = \aleph_0$

Let F be the Fréchet filter on I and U be a ultrafilter extending F]

Let $M = \prod_{i \in I} M_i / U$

Claim $M \models \Sigma$

Compactness theorem ultraproduct

Definition (Measurable cardinal)

κ is a measurable cardinal if κ is uncountable and there is a $< \kappa$-complete non-principle ultrafilter on κ

Fact

If κ is a measurable cardinal, then κ is also a weakly compact cardinal

The proof is an easy generation of the proof of compactness theorem

Compactness theorem ultraproduct

Definition (Measurable cardinal)

κ is a measurable cardinal if κ is uncountable and there is a $< \kappa$-complete non-principle ultrafilter on κ

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If κ is a measurable cardinal, then κ is also a weakly compact cardinal

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Compactness theorem ultraproduct

Definition (Measurable cardinal)

κ is a measurable cardinal if κ is uncountable and there is a $< \kappa$-complete non-principle ultrafilter on κ

Fact

If κ is a measurable cardinal, then κ is also a weakly compact cardinal

The proof is an easy generation of the proof of compactness theorem

Next on Set Theory

Descriptive set theory

Exercise

1. Let A be a set of uncountable cardinals such that for all regular cardinal λ, $A \cap \lambda$ is not stationary in λ. Show that there exists a regressive injection $f\colon A \to \mathrm{OR}$ [Hint: If λ is regular and $A \cap \lambda$ is unbounded in λ, then A' is a club in λ and disjoint from $A \cap \lambda$]