Set Theory

集合论

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

Fall 2015

Previously on Set Theory

- Stationary sets
- Is club filter a ultrafilter (Is there a pair of disjoint stationary sets)?
- Solovary theorem

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Some logic

Symbols:

- Variable: $v_1, v_2, v_3, \ldots, x, y, z, \ldots, X, Y, Z, \ldots$
- Connective: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$
- Quantifier: $\forall, \exists$
- Predicate: $=, \in$
- Parentheses: $($, $)$

Some logic

- Expressions: finite sequence of symbols
- Well-formed formula:
  - $x = y$, $x \in y$
  - $\neg\alpha$, $\alpha \vee \beta$, $\alpha \wedge \beta$, $\alpha \rightarrow \beta$, etc.
  - $\exists x \alpha$, $\forall x \alpha$

recursion definition on ω

Live

$\forall x \exists y\, x \in y$

- Occur free: E.g. If y occurs freely in α and $y \neq x$, then y occurs freely in $\forall x \alpha$
- Sentence: Formula with no free variable occurrence

Some logic

$\varphi(x_1, \ldots x_n)$ is a formula with at most $x_1, \ldots, x_n$ occurs freely in it. Let E be a binary relation on M. We define the relativization of φ into (M, E) $\varphi^{(M,E)}(x_1, \ldots, x_n)$ to be:

- $(x_i = x_j)^{(M,E)} =_{\mathrm{df}} x_i = x_j$
- $(x_i \in x_j)^{(M,E)} =_{\mathrm{df}} x_i E x_j$

  $M = \{x \mid \varphi_m(x)\}$
- $(\neg\psi)^{(M,E)} =_{\mathrm{df}} \neg\psi^{(M,E)}$
- $(\psi_1 \Box \psi_2)^{(M,E)} =_{\mathrm{df}} \psi_1^{(M,E)} \Box \psi_2^{(M,E)}$

  $\Box = \wedge, \vee, \rightarrow, \leftrightarrow$
- $(\exists y \psi(y, x_1, \ldots, x_n))^{(M,E)} =_{\mathrm{df}} \exists y(\underbrace{y \in M}_{\varphi_m(x)} \wedge \psi^{(M,E)}(y, x_1, \ldots, x_n))$

Some logic

Let E be a binary relation on M and $a_1, \ldots, a_n \in M$. We can define recursively so that

$$(m, E) \models \sigma \iff \sigma^{(m,E)}$$

$$(M, E) \models \ulcorner\varphi\urcorner(a_1, \ldots a_n) \Leftrightarrow \varphi^{(M,E)}(a_1, \ldots, a_n)$$

True $\quad \Theta(m, E, \ulcorner\varphi\urcorner, a_1 \ldots a_n)$

Some logic

Theorem (Godel's completeness theorem)

Let Σ be a set of formula. Then

$\forall x\, \varphi(x)$

$$\mathrm{Con}(\Sigma) \leftrightarrow \exists (M, E) \forall \sigma \in \Sigma (M, E) \vDash \sigma$$

EXE:

Inaccessible cardinal

$ZFC \vdash \exists! x \; x = V_\omega$

$x, y \in V_m$

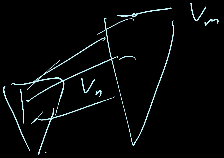
$\{x, y\} \in V_{m+1} \subseteq V_\omega$

$x \in V_m \quad x \subseteq V_m$

$\bigcup x \in V_{m+1}$

Fact (ZFC)

Con(ZFC – Axiom of Infinity)



$x \in V_n$

$\mathcal{P}(x) \in V_{n+1}$

Inaccessible cardinal

The set-theoretical operations are closed under V_ω. In particular, ω is inaccessible from below

Inaccessible cardinal

Definition (Inaccessible cardinal)

- We say κ is a weakly inaccessible cardinal if κ is an uncountable regular limit cardinal
- We say κ is a strong limit cardinal if $2^{\lambda} < \kappa$ for all $\lambda < \kappa$
- We say κ is a (strongly) inaccessible cardinal if κ is an uncountable regular strong limit cardinal

Inaccessible cardinal

Proof $\alpha \mapsto \varphi_\alpha$ is increasing ($\alpha<\beta \Rightarrow \varphi_\alpha < \varphi_\beta$)

then $\varphi_\alpha \geq \alpha$

$\varphi_\alpha = \sup\{\varphi_\xi \mid \xi < \alpha\}$. so it is sufficient to show $\varphi_\xi < \alpha$ ($\xi<\alpha$)

$\varphi_\xi < \alpha \Rightarrow \varphi_{\xi+1} < \alpha$

$\varphi_\xi < \alpha$ for $\xi < \beta$, $\beta<\alpha$ is limit, then $\varphi_\beta = \sup_{\xi<\beta} \varphi_\xi < \alpha$

Fact

- Every strong limit cardinal is limit cardinal, and so every weakly inaccessible cardinal is strongly inaccessible
- If α is weakly/strong inaccessible, then $\alpha = \aleph_\alpha$. And α is not the first such cardinal.

$\alpha_0 = \varphi_0$

$\alpha_1 = \varphi_{\alpha_0}$

$\alpha_2 = \varphi_{\alpha_1}$

$\vdots$

$\alpha_\omega = \sup_{n\in\omega} \alpha_n$

$= \sup_{n\in\omega} \varphi_{\alpha_n}$

$= \varphi_{\alpha_\omega}$

Inaccessible cardinal

Fact

If κ is inaccessible, then $V_\kappa \models ZFC$

- ZFC $+$ there exists an inaccessible cardinal $\vdash$ Con(ZFC)
- ZFC $+$ there exists a weakly inaccessible cardinal $+$ GCH holds below the first weakly inaccessible cardinal $\vdash$ Con(ZFC)

Inaccessible cardinal

Theorem (Gödel's second incompleteness theorem)

If T is a recursive theory extends $\mathrm{ZF} - \mathrm{Inf} - \mathrm{Rep} - \mathrm{Pow}$, then $T \nvdash \mathrm{Con}(T)$ unless T is inconsistent

Corollary

$\mathrm{ZFC} \nvdash$ there exists an (weakly) inaccessible cardinal

Inaccessible cardinal

Theorem (Gödel's second incompleteness theorem)

If T is a recursive theory extends $\mathrm{ZF} - \mathrm{Inf} - \mathrm{Rep} - \mathrm{Pow}$, then $T \nvdash \mathrm{Con}(T)$ unless T is inconsistent

Corollary (Con(ZFC))

$\mathrm{ZFC} \nvdash$ there exists an (weakly) inaccessible cardinal

Mahlo cardinal

Definition (Mahlo cardinal)

A cardinal κ is called Mahlo if κ is inaccessible and the set of all regular cardinal below κ is a stationary set in κ

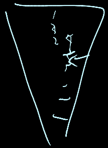
$\{\lambda < \kappa \mid \lambda \text{ is regular}\}$ is stat.

Mahlo cardinal

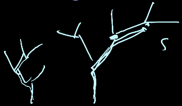
Fact if κ is inaccessible, then $C = \{\lambda < \kappa \mid \lambda \text{ is strong limit}\}$ is a club

- If κ is Mahlo, then the set of inaccessible cardinals below κ is stationary, i.e. κ is the κth inaccessible cardinal
- ZFC + there exists a Mahlo cardinal $\vdash$ Con(ZFC + there exists an inaccessible cardianl)
- ZFC + there exists an inaccessible cardianl $\nvdash$ there exists an Mahlo cardinal

there are proper class many inaccessible cardinal



Tree and weakly compact cardinal

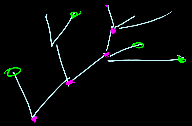


Definition

- A partially ordered set $(T, <_T)$ is a tree if for each $s \in T$ the set $\left\{ t \in T \,\middle|\, t <_T s \right\}$ is well-ordered by $<_T$
- For each $s \in T$, we define the level of s in T to be
  $\mathcal{L}_T(s) \overset{\text{otp}}{=} \left\{ t \in T \,\middle|\, t <_T s \right\}$
- Define the height of T as
  $\text{height } T = \sup\left\{ \mathcal{L}_T(s) + 1 \,\middle|\, s \in T \right\}$

Tree and weakly compact cardinal

$\mathcal{L}_T^\alpha = \{s \in T \mid \mathcal{L}_T(s) = \alpha\}$



Definition

- A set $C \subset T$ is a chain in T if for all $s, t \in C$, $s <_T t$ or $s = t$ or $t <_T s$
- A chain C is maximal if there is no chain $C' \supsetneq C$
- A chain C in T is cofinal if for all $\alpha <$ height T there is some $s \in C$ with $\mathcal{L}_T(s) \geq \alpha$
- $A \subset T$ is an antichain if for any $s, t \in A$ neither $s <_T t$ nor $s = t$ nor $t <_T s$

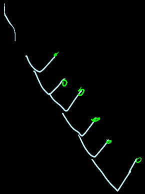
Tree and weakly compact cardinal

Definition

Let κ be a regular cardinal

- We say $(T, <_T)$ is a κ-tree if $T = \kappa$ and for each $\alpha < \kappa$, $\mathcal{L}_T^\alpha = \left\{ s \in T \,\middle|\, \mathcal{L}_T(s) = \alpha \right\}$ has cardinality $< \kappa$
- A κ-tree is a κ-Aronszajn tree if there is no cofinal chain in T
- A κ-tree is a κ-Souslin tree if it has no antichain of size κ

Tree and weakly compact cardinal



Fact

Every κ-Souslin tree is a κ-Aronszajn tree

Lemma (König)

There is no ω-Aronszajn tree

Tree and weakly compact cardinal

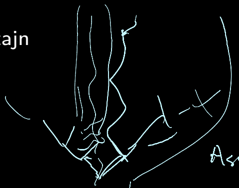
Theorem (Aronszajn)

There is an Aronszajn tree (ω_1-Aronszajn tree)

Thm. There is an ω_1-Aronszajn tree

Consider $(\omega^{<\omega_1}, \lessdot)$

where $s \lessdot t$ iff $s \subsetneq t$



Let T_0 be the subtree of $\omega^{<\omega_1}$ s.t.

$T_0 = \{ s \in \omega^{<\omega_1} \mid s \text{ is } 1\text{-}1 \}$

Then T_0 has no cofinal chain

For otherwise $\bigcup C : \omega_1 \xrightarrow{1-1} \omega$

But T_0 is not a κ-tree

[If α is infinite $\{ s \mid s \text{ is } 1\text{-}1,\ s \in \omega^{\alpha} \}$ is uncountable]

For each $\alpha < \omega_1$, define $s_\alpha \in T_0$

a) $\omega / \mathrm{ran}\, s_\alpha$ is infinite

b) For $\xi < \alpha$, $s_\alpha \restriction \xi =^* s_\xi$

where if $\mathrm{dom}\, t = \mathrm{dom}\, s$

we say $t =^* s$ iff $\{ \xi \mid t(\xi) \neq s(\xi) \}$ is finite

Assume $\{ s_\alpha : \alpha < \omega_1 \}$ is chosen,

Define $T_1 = \{ t \in T_0 \mid t =^* s_{\mathrm{dom}(t)} \}$

Then T_1 is a tree,

a ω_1-Aronszajn tree

To choose $\langle s_\alpha : \alpha < \omega_1 \rangle$

Let $s_n = \mathrm{id} \restriction n$

For $\alpha+1$, let $s_{\alpha+1} = s_\alpha \cup \{(\alpha, n)\}$ for $n \notin \mathrm{ran}\, s_\alpha$

For α is limit

Fix $\langle \beta_n : n < \omega \rangle$ is cofinal in α

For $n < \omega$, define $t_0 = s_{\beta_0}$

$t_{n+1} = t_n \cup s_{\beta_{n+1}} \restriction (\beta_{n+1} - \beta_n)$

$t_{n+1} =^* s_{\beta_{n+1}}$

$t_n =^* s_{\beta_n}$

Let $t = \bigcup_n t_n$

$t(\xi) = \begin{cases} t(\xi) & \xi \neq \beta_n \\ t(\xi) \cdot 2 & \xi = \beta_n \end{cases}$

Tree and weakly compact cardinal

Definition

- A cardinal κ has tree property if there is no κ-Aronszajn tree
- A cardinal κ is weakly compact if it is inaccessible and has tree property

Next on Set Theory

- More on weakly compact cardinal
- Measurable cardinal