Set Theory

集合论

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Previously on Set Theory

Cardinal arithmetic

- $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$
- $\mathrm{cf}(2^\kappa) > \kappa$ and $\kappa^{\mathrm{cf}\,\kappa} > \kappa$
- Assuming SCH holds below κ. Then for $\mu < \kappa$

$$\mu^\lambda = \begin{cases} 2^\lambda, & \text{if } \mu \leq 2^\lambda, \\ \mu^+, & \text{if } \mu > 2^\lambda \text{ is a limit cardinal and } \mathrm{cf}\,\mu \leq \lambda, \\ \mu, & \text{otherwise.} \end{cases}$$

Previously on Set Theory

Club

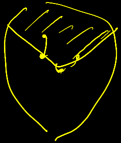
- A club in α is a closed and unbounded set in α
- Clubs in α are nontrivial if $\operatorname{cf}\alpha > \omega$

  C is club $\Rightarrow$ C' is club
- Clubs in α are closed under "prime operator",

  $< \operatorname{cf}\alpha$-intersection, and diagonal intersection.

  $\triangle_{\xi<\alpha} C_\xi$

Previously on Set Theory



Filter

- Filter gives a measure of bigness of sets
- $< \delta$-closed filter, ultrafilter
- Club filter on α: $\left\{X \subset \alpha \,\middle|\, \text{there is a club } C \subset X\right\}$

How is the club filter looks like?

Club filter

Fact

Assume $\mathrm{cf}\,\alpha > \omega$. The club filter on α is $< \mathrm{cf}\,\alpha$ closed

Stationary sets

Definition

Let $X \neq \emptyset$ and F be a filter on X. We say a set $A \subset X$ is **positive with respect to F** if $A \cap B \neq \emptyset$ for every $B \in F$

A set $S \subset \alpha$ is **stationary** if it is positive with respect to the club filter on α. That is S intersects every club in α

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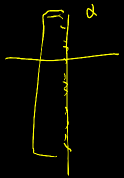
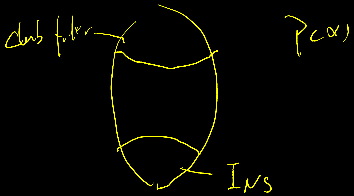
A set $S \subset \alpha$ is stationary if it is positive with respect to the club filter on α. That is S intersects every club in α

Fact

$(S \cap C) \cap C' = S \cap (C \cap C')$



- If S is a stationary set on α and C is a club on α, then $S \cap C$ is stationary on α
- If $N \subset \alpha$ is non-stationary, then $\alpha \setminus N$ contains a club ∈ club filter
- If S is stationary on a uncountable regular cardinal κ, then card $S = \kappa$

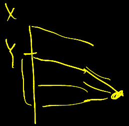


Stationary sets

Definition

$F \subseteq \mathcal{P}(\kappa)$

Let κ be a regular cardinal. A filter F on κ is normal if for every function $f : \kappa \to \kappa$ that is regressive on a positive set X, i.e. $f(\xi) < \xi$ for $\xi \in X$, then f is constant on a positive set.



$\exists$ positive $Y \subseteq X$ and γ s.t.

$\forall \xi \in Y, \ f(\xi) = \gamma$

Stationary sets

Lemma

Let κ be regular and F be a filter on κ. The following are equivalent.

1. F is normal
2. F is closed under diagonal intersection, i.e. $\Delta_{\xi<\kappa} A_\xi \in F$ for every sequence $\langle A_\xi : \xi < \kappa \rangle$ of elements in F.

normal ⇔ closed under diagonal intersection

(⇒) Let $C = \triangle_{\xi<\kappa} C_\xi$

where $C_\xi \in F$ $(\xi<\kappa)$

Assume $C \notin F$. Then $\bar{C}$ is positive

[Let $A \in F$, if $A \cap \bar{C} = \emptyset$, then $A \subset C$, so $C \in F$ ⊣]

$\bar{C} = \{\alpha<\kappa \mid \exists \xi<\alpha \ \alpha \notin C_\xi\}$

For each $\alpha \in \bar{C}$, let ξ_α be the least s.t. $\alpha \notin C_{\xi_\alpha}$

$\alpha \mapsto \xi_\alpha$ is a regressive function on $\bar{C}$

there is positive $D \subseteq \bar{C}$ and $\gamma<\kappa$ s.t.

$\alpha \notin C_\gamma$ for all $\alpha \in D$

i.e. $D \cap C_\gamma = \emptyset$ ⊣

(positive) ($\in F$)

(⇐) Fix positive $X \subseteq \kappa$ and fix regressive on X, i.e. $f(\xi)<\xi$ for $\xi \in X$.

We find positive $Y \subseteq X$ s.t. $f(\xi)=\gamma$ for $\xi \in Y$

Let $Y_\gamma = \{\xi<\kappa \mid f(\xi)=\gamma\}$ $(\gamma<\kappa)$

Assume Y_γ is not positive, then

there is $Z_\gamma \in F$ s.t. $Y_\gamma \cap Z_\gamma = \emptyset$

Let $Z = \triangle_{\gamma<\kappa} Z_\gamma \in F$

Then for each η, $\eta \in Z \iff$

$\eta \in Z_\gamma$ for $\gamma<\eta$, i.e.

i.e. $\eta \notin Y_\gamma$ ...

i.e. $f(\eta) \geq \eta$, ⊣ $\eta \in X$.

Thus $X \cap Z = \emptyset$ ⊣ □

Corollary

For each regular κ, the club filter on κ is normal.

Stationary sets

Definition

Let κ be an uncountable regular cardinal, $\lambda < \kappa$. Define

$$E^\kappa_\lambda = \left\{\alpha < \kappa \,\middle|\, \operatorname{cf}\alpha = \lambda\right\}$$

Fact

- E^κ_λ is stationary
- If $\kappa \geq \aleph_2$, then the club filter on κ is not an ultrafilter

Stationary sets

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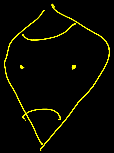

Fact

- E^κ_λ is stationary $E^\kappa_{\omega_0}$, $E^\kappa_{\omega_1}$,
- If $\kappa \geq \aleph_2$, then the club filter on κ is not an ultrafilter

Stationary sets

Lemma

Let κ be an uncountable regular cardinal and $\lambda < \kappa$. Every stationary subset of E^{κ}_{λ} is the union of κ disjoint stationary sets.

Stationary sets

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Let κ be an uncountable regular cardinal and $\lambda < \kappa$. Every stationary subset of E^{κ}_{λ} is the union of κ disjoint stationary sets.

$S \subseteq E^\kappa_\lambda$ is the union of κ disjoint stationary sets

Proof. Fix $S \subseteq E^\kappa_\lambda$.

For each $\alpha \in S$, fix $\langle \beta^\alpha_\xi : \xi < \lambda \rangle$ a cofinal sequence in α

Claim. there exists $\xi < \lambda$ s.t. for each $\eta < \kappa$,

$S^\eta = \{\alpha \mid \beta^\alpha_\xi > \eta\}$ is stationary

Assume not. i.e. for each $\xi < \lambda$, there is η_ξ s.t. there is club C_ξ s.t. for each $\alpha \in C_\xi$ $\beta^\alpha_\xi \leq \eta_\xi$

Let $\eta = \sup_{\xi < \lambda} \eta_\xi$, then $\eta < \kappa$,

Let $C = \bigcap_{\xi < \lambda} C_\xi$, then C is club.

For each $\alpha \in C$, for each $\xi < \lambda$

$\beta^\alpha_\xi < \eta_\xi < \eta$

i.e. C is bounded by η

Fix that ξ.

Let $f : S \to \kappa$ be s.t. $f(\alpha) = \beta^\alpha_\xi$

Then f is regressive on S^η

Then there is $S_\eta \subseteq S^\eta$ s.t.

$f(\alpha) = \gamma_\eta$ for each $\alpha \in S_\eta$

Note $\kappa > \gamma_\eta > \eta$

$\eta \to \gamma_\eta$

$|\{\gamma_\eta \mid \eta < \kappa\}| = \kappa$

And if $\gamma_{\eta_1} \neq \gamma_{\eta_2}$, then $S_{\eta_1} \cap S_{\eta_2} = \phi$

□

Stationary sets

F is club filter on ω_1,
Then $E^{\omega_1}_{\omega}$ is stationary on ω_1,

Corollary

Let κ be an uncountable regular cardinal. The club filter on κ is not an ultrafilter

Stationary sets

$E^{\kappa}_{\lambda} \cap S$

Theorem (Solovay)

Let κ be an uncountable regular cardinal. Every stationary subset of κ is the union of κ disjoint stationary sets.

Stationary sets

Fact

Fix $D_s = \{\alpha < \kappa \mid \operatorname{cf}\alpha < \alpha\}$, $D_r = \{\alpha < \kappa \mid \operatorname{cf}\alpha = \alpha\}$, and S is a stationary set. Then

$\kappa \setminus S \cup \kappa \setminus D_s = D_r \cup \kappa \setminus S$

$C = \kappa \setminus (S \cap D_s)$. Then $D_r \cap S = (D_r \cup \kappa \setminus S) \cap S$ (club)

- Either $S \cap D_s$ or $S \cap D_r$ is stationary on $\kappa \setminus$ stationary
- If $S \cap D_s$ is stationary, then $S \cap D_s$ $(= S')$ can be split into κ many stationary sets

$\alpha \to \operatorname{cf}\alpha$ is regressive on S, so there is λ and stat. $S_\lambda \subseteq S$ s.t. cf is constant on S_λ $(\forall \alpha \in S_\lambda \ \operatorname{cf}\alpha = \lambda)$ i.e. $S_\lambda \subseteq E^\kappa_\lambda$

Stationary sets

Fact

Fix $D_s = \left\{\alpha < \kappa \,\middle|\, \operatorname{cf}\alpha < \alpha\right\}$, $D_r = \left\{\alpha < \kappa \,\middle|\, \operatorname{cf}\alpha = \alpha\right\}$, and S is a stationary set. Then

- Either $S \cap D_s$ or $S \cap D_r$ is stationary
- If $S \cap D_s$ is stationary, then $S \cap D_s$ can be split into κ many stationary sets

Stationary sets

Thus, it is left to consider the case when $S \cap D_r$ is stationary

Lemma

κ is an uncountable regular cardinal. Let S be a stationary subset of κ, and $S \subset D_r$. Then the set

$$T = \left\{ \alpha \in S \,\middle|\, S \cap \alpha \text{ is not stationary in } \alpha \right\}$$

is stationary in κ

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$T =$

$\{\alpha \in S \mid S \cap \alpha \text{ is not stationary in } \alpha\}$

is stationary

Fix club C.

Then $C' \subseteq C$ is club on κ.

Then $C' \cap S \neq \emptyset$.

Let α be the least in $C' \cap S$

Since $\alpha \in C'$, $C \cap \alpha$ is club in α

So $C' \cap \alpha = (C \cap \alpha)' - \{\alpha\}$

But $(C' \cap \alpha) \cap (S \cap \alpha) = \emptyset$

[α is the least]

So $S \cap \alpha$ is not stat. in α

Therefore $\alpha \in T \cap C$

Now it is left to show that:

T can be split into κ many stationary sets

For each $\alpha \in T$, since $T \cap \alpha$ is not stat.

Let $\langle \beta^\xi_\alpha : \xi < \alpha \rangle$ be unbounded increasing and continuous

s.t. each $\beta^\alpha_\xi \notin T$

Claim there $\xi < \kappa$, for each $\eta < \kappa$,

$\{\alpha \mid \beta^\alpha_\xi > \eta\}$ is stat.

Assume not: for each $\xi < \kappa$, there is $\eta_\xi < \kappa$

and C_ξ s.t. for each $\alpha \in C_\xi$, $\beta^\alpha_\xi \le \eta_\xi$

Let $C = \triangle_{\xi<\kappa} C_\xi$

For each $\alpha \in C$, for all $\xi < \alpha$

$\beta^\alpha_\xi \le \eta_\xi$

$D = \{\alpha \mid \alpha \ge \sup_{\xi<\alpha} \eta_\xi\}$

$= \triangle_{\xi<\kappa} \{\alpha \mid \alpha > \eta_\xi\}$ is club

Consider $\gamma < \alpha$ s.t.

$\gamma, \alpha \in C \cap D \cap T$

For each $\xi < \gamma$

$\beta^\alpha_\xi \le \eta_\xi \le \sup_{\theta<\gamma} \eta_\theta \le \gamma$ ($\gamma \in D$)

(since $\alpha \in C$)

Since $\xi \to \beta^\alpha_\xi$ is increasing, so

$\sup_{\xi<\gamma} \beta^\alpha_\xi = \gamma \in T$

? $\quad \| $

$\beta^\alpha_\gamma \notin T$

The rest is the same

Next on Set Theory

Some small large cardinals

Exercise

1. Show that $(\prod_{i \in I} \alpha_i)^\kappa = \prod_{i \in I}(\alpha_i^\kappa)$
2. Let κ be a uncountable regular cardinal. Show that if $S \subset \kappa$ is unbounded, then S' is a club in κ
3. $(*)$ Let $S \subset \omega_1$ be stationary, and let $\alpha < \omega_1$. Show that S has a closed subset of order type $\alpha + 1$