

Set Theory

集合论

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

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Previously on Set Theory

Cofinality

- $\text{cf } \alpha = \min \{ \beta \in \text{OR} \mid \text{there is a cofinal } f: \beta \rightarrow \alpha \}$
- regular / singular cardinal
- $\text{cf } \alpha$ is regular cardinal for each infinite α
- Every infinite successor cardinal is regular

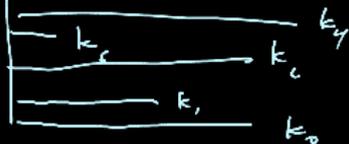
Previously on Set Theory

Some facts on cardinal exponentiation

- $2^\kappa = (2^{<\kappa})^{\text{cf } \kappa}$
- Assume κ be infinite singular cardinal, and for all μ such that $\mu_0 \leq \mu < \kappa$, $2^\mu = \lambda$. Then $2^\kappa = \lambda$
- $(\kappa^+)^{\lambda} = \kappa^{\lambda} \cdot \kappa^+$ (Hausdorff Theorem)

α

Infinite sum and infinite production



Let I be an index set and for each $i \in I$, κ_i is a cardinal. We define

$$\sum_{i \in I} \kappa_i = \text{card} \left(\bigcup_{i \in I} (\kappa_i \times \{i\}) \right)$$

and

$$\prod_{i \in I} \kappa_i = \text{card} \left\{ g \in \left(\bigcup_i \kappa_i \right)^I \mid (\forall i \in I) g(i) \in \kappa_i \right\}$$



Infinite sum and infinite production

$$\kappa < 2^\kappa$$

Theorem (König)

Let $I \neq \emptyset$. Assume that $\kappa_i < \lambda_i$ for all $i \in I$. Then

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$$

König Theorem: Assume $I \neq \emptyset$ and

$\kappa_i > \lambda_i$.

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$$

Assume $\sum_{i \in I} \kappa_i = \kappa \geq \prod_{i \in I} \lambda_i$

Let $\langle g_{(\eta, i)} : \eta \in \kappa_i \rangle_{i \in I}$ be the enumeration
of $\prod_{i \in I} \lambda_i = \{g \mid \forall i \in I, g(i) \in \lambda_i\}$

We want to find $f: I \rightarrow \bigcup_{i \in I} \lambda_i$ s.t.

$f \neq g_s$ for all s

For each $i \in I$

$$\left\{ g_{(\eta, i)}(i) \mid \eta \in \kappa_i \right\} \left| \leq \kappa_i \right.$$

Thus $\lambda_i \setminus \{g_{(\eta, i)}(i) \mid \eta \in \kappa_i\} \neq \emptyset$

Let $f(i)$ be the least in that set

So $f \neq g_{(\eta, i)}$ for $\eta \in \kappa_i$

□

Infinite sum and infinite production

Part

1, Assume $\langle \alpha_i \mid i < \kappa \rangle$ is cofinal in \mathbb{Z}^κ

$$\sum_{i < \kappa} \alpha_i \geq \mathbb{Z}^\kappa = \kappa^\kappa = \prod_{i < \kappa} \kappa$$

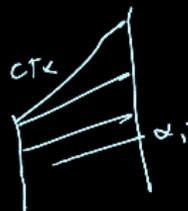
But, $\kappa > \alpha_i \Rightarrow$

Corollary

For all infinite cardinals κ , $\text{cf}(2^\kappa) > \kappa$ and $\kappa^{\text{cf} \kappa} > \kappa$

2,

$$\kappa^{\text{cf} \kappa} = \prod_{i < \text{cf} \kappa} \kappa > \sum_{i < \text{cf} \kappa} \alpha_i \geq \kappa$$



GCH

$$2^k = k^+$$

Singular cardinal hypothesis

Definition (Singular cardinal hypothesis)

Singular cardinal hypothesis (SCH) is the statement that for all singular limit cardinal κ ,

$$\kappa^{\text{cf } \kappa} = 2^{\text{cf } \kappa} \cdot \kappa^+$$

Singular cardinal hypothesis

Fact

- $\kappa^{\text{cf}\kappa} \geq 2^{\text{cf}\kappa} \cdot \kappa^+$ holds for all infinite cardinal κ
 $\kappa^\kappa \geq \sum^\kappa = 2^{\text{cf}\kappa} \geq \kappa^+$ $\text{cf}\kappa = \kappa$
- $\kappa^{\text{cf}\kappa} = 2^{\text{cf}\kappa} \cdot \kappa^+$ is true for every infinite regular cardinal
- GCH implies SCH

$$\kappa^+ = \sum^\kappa = \kappa^\kappa \geq \kappa^{\text{cf}\kappa}$$

Singular cardinal hypothesis

Lemma

Let κ be a limit cardinal and SCH holds below κ . Then for every infinite $\mu < \kappa$ and every infinite λ

$$\mu^\lambda = \begin{cases} 2^\lambda, & \text{if } \mu \leq 2^\lambda, \\ \mu^+, & \text{if } \mu > 2^\lambda \text{ is a limit cardinal and } \text{cf } \mu \leq \lambda, \\ \mu, & \text{otherwise.} \end{cases}$$

By induction on $\mu < \kappa$, we show

$$\mu^\lambda = \begin{cases} 2^\lambda, & \text{if } \mu \leq 2^\lambda, \\ \mu^+, & \text{if } \mu > 2^\lambda \text{ is a limit cardinal and } \text{cf } \mu \leq \lambda, \\ \mu, & \text{otherwise.} \end{cases}$$

Case 1. $\mu \leq 2^\lambda$

Case 2. $\mu > 2^\lambda$ and $\mu = \nu^+$ is
successor

By induction on $\mu < \kappa$, we show

$$\mu^\lambda = \begin{cases} 2^\lambda, & \text{if } \mu \leq 2^\lambda, \\ \mu^+, & \text{if } \mu > 2^\lambda \text{ is a limit cardinal and } \text{cf } \mu \leq \lambda, \\ \mu, & \text{otherwise.} \end{cases}$$

Case 1. $\mu \leq 2^\lambda$

$$\mu^\lambda \leq (2^\lambda)^\lambda = 2^{\lambda \cdot \lambda} = 2^\lambda$$

Case 2. $\mu > 2^\lambda$ and $\mu = \nu^+$ is
successor

By induction on $\mu < \kappa$, we show

$$\mu^\lambda = \begin{cases} 2^\lambda, & \text{if } \mu \leq 2^\lambda, \\ \mu^+, & \text{if } \mu > 2^\lambda \text{ is a limit cardinal and } \text{cf } \mu \leq \lambda, \\ \mu, & \text{otherwise.} \end{cases}$$

Case 1. $\mu \leq 2^\lambda$

Case (2.1) $\mu > 2^\lambda$ and $\mu = \nu^+$ is

successor

either $\nu > 2^\lambda$, $\overset{\text{by IH}}{\nu}^\lambda = \nu^+$ or $\nu^\lambda = \nu$

By Hausdorff theorem $\mu^\lambda = (\nu^+)^\lambda = \nu^\lambda \cdot \nu^+ = \nu^+$

or $\nu = 2^\lambda$, by IH, $\nu^\lambda = 2^\lambda = \nu$ \nearrow

By induction on $\mu < \kappa$, we show

$$\mu^\lambda = \begin{cases} 2^\lambda, & \text{if } \mu \leq 2^\lambda, \\ \mu^+, & \text{if } \mu > 2^\lambda \text{ is a limit cardinal and } \text{cf } \mu \leq \lambda, \\ \mu, & \text{otherwise.} \end{cases}$$

Case 3. $\mu > 2^\lambda$

Case 3.1. $\lambda < \text{cf } \mu$

Case 3.2. $\text{cf } \mu \leq \lambda$

$$\mu^+ \leq \mu^{\text{cf } \mu} \leq \mu^\lambda \leq \left(\prod_{i < \text{cf } \mu} \alpha^i \right)^\lambda = \prod_{i < \text{cf } \mu} \alpha_i^\lambda \leq \prod_{i < \text{cf } \mu} \alpha_i^+ \leq \prod_{i < \text{cf } \mu} \mu = \mu^{\text{cf } \mu} = 2^{\text{cf } \mu} \cdot \mu^+ = \mu^+$$

By induction on $\mu < \kappa$, we show

$$\mu^\lambda = \begin{cases} 2^\lambda, & \text{if } \mu \leq 2^\lambda, \\ \mu^+, & \text{if } \mu > 2^\lambda \text{ is a limit cardinal and } \text{cf } \mu \leq \lambda, \\ \mu, & \text{otherwise.} \end{cases}$$

Case 3. $\mu > 2^\lambda$

Case 3.1. $\lambda < \text{cf } \mu$

$\langle \mu_i : i < \text{cf } \mu \rangle$ is cofinal in μ

by IH

$$\mu^\lambda \leq \sum_{i < \text{cf } \mu} \mu_i^\lambda \leq \sum_{i < \text{cf } \mu} \mu_i^+ \leq \text{cf } \mu \cdot \mu = \mu \leq \mu^\lambda$$

Case 3.2. $\text{cf } \mu \leq \lambda$

$$\mu^+ \leq \mu^{\text{cf } \mu} \leq \mu^\lambda \leq \left(\prod_{i < \text{cf } \mu} \alpha_i^\lambda \right)^\lambda = \prod_{i < \text{cf } \mu} \alpha_i^\lambda \leq \prod_{i < \text{cf } \mu} \alpha_i^+ \leq \prod_{i < \text{cf } \mu} \mu = \mu^{\text{cf } \mu} = 2^{\text{cf } \mu} \cdot \mu^+ = \mu^+$$

By induction on $\mu < \kappa$, we show

$$\mu^\lambda = \begin{cases} 2^\lambda, & \text{if } \mu \leq 2^\lambda, \\ \mu^+, & \text{if } \mu > 2^\lambda \text{ is a limit cardinal and } \text{cf } \mu \leq \lambda, \\ \mu, & \text{otherwise.} \end{cases}$$

Case 2

~~Case 2~~ $\mu > 2^\lambda$

Case 2.1 $\lambda < \text{cf } \mu$

$\langle \alpha_i : i < \text{cf } \mu \rangle$ is cofinal in μ

$\mu \leq \sum_{i < \text{cf } \mu} \alpha_i < \prod_{i < \text{cf } \mu} \alpha_i$

$\text{cf } \mu \leq \lambda$
 $2^\lambda \leq \mu$
 $2^{\text{cf } \mu} \leq 2^\lambda \leq \mu$

Case 3 $\text{cf } \mu \leq \lambda$

$$\mu^+ \leq \mu^{\text{cf } \mu} \leq \mu^\lambda \leq \left(\prod_{i < \text{cf } \mu} \alpha_i \right)^\lambda = \prod_{i < \text{cf } \mu} \alpha_i^\lambda \stackrel{\text{IH}}{\leq} \prod_{i < \text{cf } \mu} \alpha_i^+ \stackrel{\alpha_i^+ \leq \mu}{\leq} \prod_{i < \text{cf } \mu} \mu \stackrel{\text{SCH}}{=} \mu^{\text{cf } \mu} = 2^{\text{cf } \mu} \cdot \mu^+ = \mu^+$$

$\mu \leq \mu^{\text{cf } \mu}$

∞

Closed unbounded set

Definition

Given a set $A \subset \text{OR}$. We say

$$\forall \theta < \xi \exists \eta \in A (\theta < \eta < \xi)$$



- ξ is a **limit point** of A , if $\xi = \sup(A \cap \xi)$
- A is **closed** if A contains all its limit points
- A is **closed in α** if A contains all its limit points less than α
- A is **unbounded in α** if α is a limit point of A
- A is a **club** in α if A is closed and unbounded in α

To see C' is closed Closed unbounded set

$A \subseteq C' \subseteq C$, Assume $\sup A < \alpha$, since C is club

Fact $\underline{C \subseteq \alpha}$ thus $\sup A \in C$, and it is a limit point so $\sup A \in C'$
 $\in C$

Let C' be the set of limit points of C .

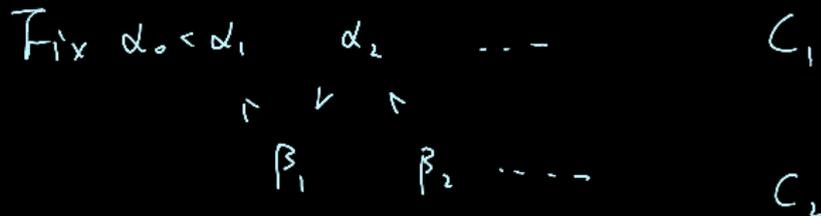
$$C' \cap \alpha \subseteq C \cap \alpha$$

- Assume $\text{cf } \alpha > \omega$. If C is a club in α , then the set of all limit points of C' is also a club in α .
Fix $\alpha_0 < \alpha$, since C is unbounded $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \dots \in C$
- If C_1 and C_2 are clubs in α , then $C_1 \cap C_2$ is also a club in α .
Then $\alpha_\omega = \sup_{\text{new}} \alpha_n < \alpha$ ($\text{cf } \alpha > \omega$), and $\alpha_\omega \in C'$
- If $\beta < \text{cf } \alpha$, and $\langle C_\xi : \xi < \beta \rangle$ is a sequence of clubs in α , then $\bigcap_{\xi < \beta} C_\xi$ is a club of α .

$C_1 \cap C_2$ is club

$\forall C_1, C_2$ is closed, $\overline{\text{intersect}}$

$\Rightarrow C_1 \cap C_2$ is unbounded in α



$$\beta = \sup \langle \alpha_0, \alpha_1, \beta_1, \alpha_2, \beta_2, \dots \rangle$$

then $\beta < \alpha$ (since cf $\alpha > \omega$), And

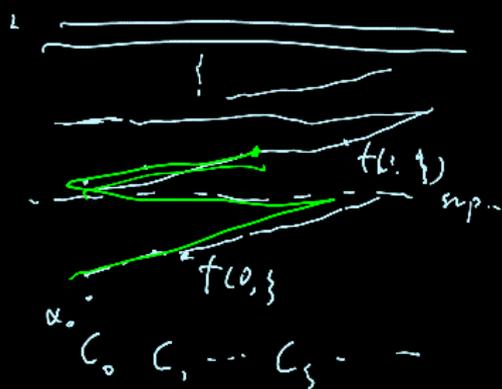
$$\begin{aligned} \beta &= \sup \langle \alpha_0, \alpha_1, \alpha_2, \dots \rangle \\ &= \sup \langle \beta_1, \beta_2, \beta_3, \dots \rangle \end{aligned} \quad \text{So } \beta \in C_1 \cap C_2$$

$\bigcap_{\xi < \beta} C_\xi$ is club

$f(n, \xi)$

Let $f(0, 0) = \alpha_0$

$f(n, \xi) =$ the least γ in C_ξ s.t. $\gamma \geq \underline{\sup} \{ f(i, \eta) \mid \eta \in \{i, i+1\} \}$



Filter

Let $X \neq \emptyset$ be a set. We say $F \subset P(X)$ is a **filter** on X if

- 1 $F \neq \emptyset$
- 2 If $A, B \in F$, then $A \cap B \in F$
- 3 If $A \in F$ and $A \subset B$, then $B \in F$



A filter F is **non-trivial** if $\emptyset \notin F$. F called an **ultrafilter** if for each $A \subset X$, either $A \in F$ or $X \setminus A \in F$.

Filter

Example



- Take $a \in X$. Then $\{A \subset X \mid a \in A\}$ is a filter on X
- $\{A \subset \omega \mid \omega \setminus A \text{ is finite}\}$ is a filter on ω
cofinite
- Let α be a limit ordinal, then

$$\{A \subset \alpha \mid \exists \beta < \alpha (\alpha \setminus \beta) \subset A\}$$



is a filter on α . It is called the **Fréchet filter** (on α)

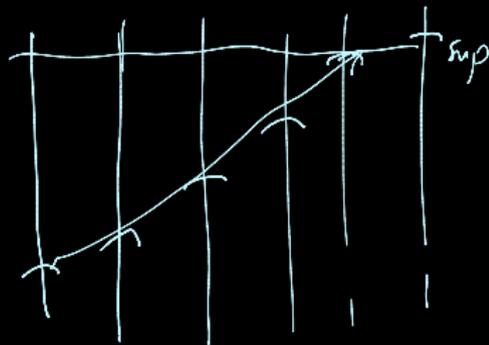
Filter

Definition

Let F be a filter on X , μ is a cardinal. We say F is $< \mu$ -closed if for each μ sequence $\langle X_\xi : \xi < \mu \rangle$ of sets in F , $\bigcap_{\xi < \mu} X_\xi \in F$

Fact

Fréchet filter (on α) is $< \text{cf } \alpha$ closed



Filter

Fact

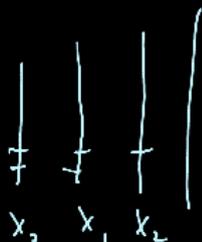
Let α be an ordinal such that $\text{cf } \alpha > \omega$. Define

$$F_\alpha = \left\{ A \subset \alpha \mid \begin{array}{l} \text{club filter} \\ \text{there is a club } C \text{ such that } C \cap \alpha \subset A \end{array} \right\}$$

Then F_α is a non-trivial $< \text{cf } \alpha$ -closed filter on α

Closed unbounded set

Definition



Let κ be regular and $\langle X_\xi : \xi < \alpha \rangle$ be a sequence of subsets of κ . Define the **diagonal intersection** of the sequence to be

$$\Delta_{\xi < \alpha} X_\xi = \left\{ \eta < \kappa \mid \eta \in \bigcap_{\xi < \eta} X_\xi \right\}$$

Closed unbounded set

Lemma

Let κ be an uncountable regular cardinal, $\langle C_\xi : \xi < \kappa \rangle$ be a sequence of clubs in κ . Then $\Delta_{\xi < \kappa} C_\xi$ is a club in κ .

$\Delta_{\xi < \kappa} C_\xi$ is a club in κ

(1) closed Let $C = \Delta_{\xi < \kappa} C_\xi$, and $A \in C$, $\sup A < \kappa$.

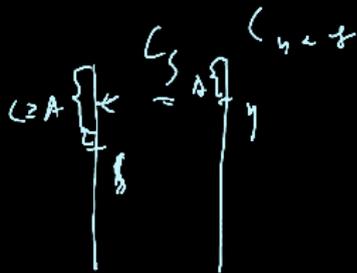
Let $\gamma = \sup A$. Note $\gamma \in C \Leftrightarrow \gamma \in \bigcap_{\xi < \gamma} C_\xi$

Fix $\xi < \gamma$

For each $\beta \in A \cap \xi$, if $\beta > \xi$, then $\beta \in C_\xi$

Then $\sup(A \cap \xi \cap C_\xi) = \gamma \in C_\xi$

i.e. $\gamma \in \bigcap_{\xi < \gamma} C_\xi$ i.e. $\gamma \in C = \Delta_{\xi < \kappa} C_\xi$



(2) unbounded Fix α_0

Find an increasing $\langle \alpha_i^* : i < \omega \rangle$ s.t. $\alpha_i^* \in C_i$

Let $\alpha_1 = \sup\{\alpha_i^* : i < \omega\}$

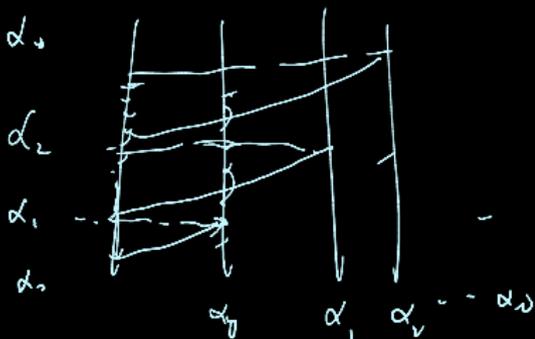
Inductively, let $\alpha_{n+1} = \sup\{\alpha_i^* : i < \omega_n\}$

where $\alpha_0^* = \alpha_0$, $\langle \alpha_i^* : i < \omega_n \rangle$ is an increasing sequence s.t.

$\alpha_i^* \in C_i$

Let $\alpha_\omega = \sup\{\alpha_n : n < \omega\}$

Check: α_ω is a limit point of each C_i ($i < \omega_n < \alpha_\omega$) \square



Next on Set Theory

Stationary sets

Exercise

1. Show that every filter can be extended to an ultrafilter (AC)
2. Let $\alpha \in \mathcal{OR}$. Show that there is a club $C \subset \alpha$ such that
 $\text{card } C = \text{otp } C = \text{cf } \alpha$