Set Theory

集合论

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Previously on Set Theory

Cardinal

- An ordinal α is a cardinal if $\alpha = \operatorname{card} \alpha = \min \left\{ \beta \,\middle|\, \beta \sim \alpha \right\}$
- Successor cardinal, limit cardinal
- enumeration of infinite cardinals: $\aleph_\alpha$

Previously on Set Theory

Cardinal arithmetic

- $\kappa + \lambda$, $\kappa \cdot \lambda$, κ^λ
- Canonical well ordering of $\mathrm{OR} \times \mathrm{OR}$ $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$
- $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$

Cardinal arithmetic

Facts of cardinal exponentiation

- $\kappa^{\lambda+\mu} = \kappa^{\lambda} \cdot \kappa^{\mu}$
- $(\kappa^{\lambda})^{\mu} = \kappa^{\lambda \cdot \mu}$

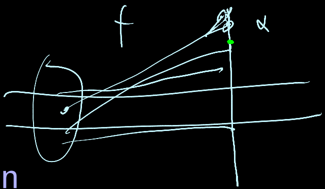
  $2^{\lambda} \leq \kappa^{\lambda} \leq (2^{\kappa})^{\lambda} \leq (2^{\lambda})^{\lambda} = 2^{\lambda \cdot \lambda} = 2^{\lambda}$
- Assume $\kappa \leq \lambda$ are infinite cardinal, then $\kappa^{\lambda} = 2^{\lambda}$
- $2^{\lambda} > \lambda$

Cardinal arithmetic

By forcing arguments, we can hardly infer anything about 2^λ for regular λ within ZFC. However, there are something we can say about κ^λ if λ is irregular, or $\kappa \geq \lambda$.

This is where we need the concept of cofinality

Cofinality



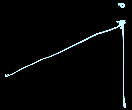
Definition

- Given $\alpha \in \mathrm{OR}$, we say function $f : A \to \alpha$ is cofinal in α if for all $\beta < \alpha$ there exists $a \in A$ such that $f(a) \geq \beta$.
- The cofinality of α (written $\mathrm{cf}\,\alpha$) is the defined as

$$\mathrm{cf}\,\alpha = \min \left\{ \beta \in \mathrm{OR} \;\middle|\; \text{there is a cofinal } f : \beta \to \alpha \right\}$$

Cofinality

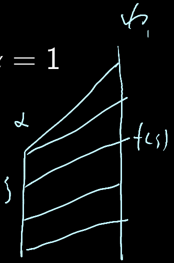
Fact $\mathrm{cf}\,\alpha \le \alpha$



Example

- If $\alpha = \beta + 1$ is a successor ordinal, then $\mathrm{cf}\,\alpha = 1$
- $\mathrm{cf}\,\omega = \mathrm{cf}(\omega + \omega) = \mathrm{cf}\,\aleph_\omega = \aleph_{\aleph_\omega} = \omega$

  $1 \to \omega_1$, $0 \to \omega_0$
- $\mathrm{cf}\,\aleph_1 = \aleph_1$



Assume $\alpha < \omega_1$ and $\mathrm{cf}\,\omega_1 = \alpha$, Let f witness that

For each $\xi < \alpha$, $f(\xi) < \omega_1$, and $\omega_1 \le \mathrm{card}\left(\bigcup_{\xi<\alpha} f(\xi)\right) \le \mathrm{card}\,|\alpha| \cdot \omega_0 < \omega_1$.

Cofinality

Assume α is limit, and f_0 witness $\mathrm{cf}\,\alpha = \beta$

Let $X = \mathrm{ran}\, f_0 \subseteq \alpha$

Let f be the enumeration of X (by Mostowski collapse)

Fact

If $\mathrm{cf}\,\alpha = \beta$, then it is witnessed by an strictly increasing function $f : \beta \to \alpha$.

Cofinality

Definition

Let α be an ordinal. We say α is regular if $\operatorname{cf} \alpha = \alpha$; otherwise, we say α is singular

Example

Cofinality

Definition

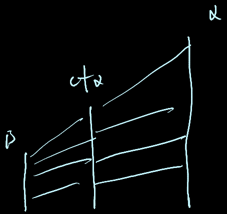
Let α be an ordinal. We say α is **regular** if $\operatorname{cf}\alpha = \alpha$; otherwise, we say α is **singular**

Example

regular: $\aleph_0$ $\aleph_1$, ,

singular: $\aleph_\omega$, $\aleph_{\omega+\omega}$,

Cofinality



Lemma

For every ordinal α, $\mathrm{cf}\,\alpha$ is regular

Cofinality

Lemma

If α is a regular ordinal, then α is a cardinal

Corollary

For every ordinal α, $\mathrm{cf}\,\alpha$ is a regular cardinal

Cofinality

Lemma

If α is a regular ordinal, then α is a cardinal

Corollary

For every ordinal α, $\operatorname{cf}\alpha$ is a regular cardinal

Cofinality

Lemma

Every successor cardinal is regular

Cofinality

Definition

- $\lambda^{<\kappa} = \sup_{\mu<\kappa} \lambda^{\mu}$

Fact

$2^{<\kappa} = \mathrm{card}(\bigcup_{i\in I} P(\alpha_i))$ where $\langle \alpha_i : i \in I \rangle$ is cofinal in κ

Cofinality

Let $\lambda = \mathrm{card}(\bigcup_{i \in I} P(\alpha_i))$

($\leq$) it is sufficient to show $2^\mu \leq \lambda$ for all $\mu < \kappa$

By cofinal, find $\alpha_j \geq \mu$. then $2^\mu \leq |P(\alpha_j)| \leq |\bigcup_{i \in I} P(\alpha_i)|$

($\geq$) Since $|I| \leq \kappa$, and $|P(\alpha_i)| \leq 2^{<\kappa}$

$$2^{<\kappa} \geq 2^{<\kappa} \cdot \kappa \geq |\bigcup_{i \in I} P(\alpha_i)|$$

Definition

- $\lambda^{<\kappa} = \sup_{\mu<\kappa} \lambda^\mu$

Fact

$2^{<\kappa} = \mathrm{card}(\bigcup_{i \in I} P(\alpha_i))$ where $\langle \alpha_i : i \in I \rangle$ is cofinal in κ

Proof

(≥) $(2^{<\kappa})^{\mathrm{cf}\kappa} \le (2^{\kappa})^{\mathrm{cf}\kappa} = 2^{\kappa\cdot\mathrm{cf}\kappa} = 2^{\kappa}$

Cofinality

$\sup_{\mu<\kappa} 2^{\mu\cdot\mathrm{cf}(\kappa)} \le 2^{\kappa}$

(≤) Let $\langle \kappa_i : i<\mathrm{cf}\kappa \rangle$ be a cofinal sequence of κ

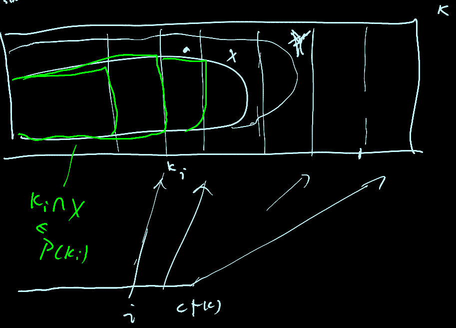
Then $2^{<\kappa} = \left| \bigcup_{i<\mathrm{cf}\kappa} 2^{\kappa_i} \right|$

It is suffice to show $|P(\kappa)| = \left|\left(\bigcup_{i<\mathrm{cf}\kappa} 2^{\kappa_i}\right)^{\mathrm{cf}\kappa}\right|$

We define $F : P(\kappa) \to \left(\bigcup_{i<\mathrm{cf}\kappa} 2^{\kappa_i}\right)^{\mathrm{cf}\kappa}$

Lemma

$$2^{\kappa} = (2^{<\kappa})^{\mathrm{cf}\,\kappa}$$

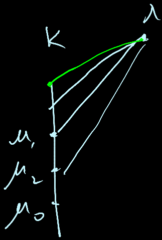


For $X \subseteq \kappa$, let $F(X)$ be the function f_X

s.t. $f_X(i) = X \cap \kappa_i$

We show F is 1-1 : For $X \neq Y$, there exist $i \in I$ s.t. $X \cap \kappa_i \neq Y \cap \kappa_i$ (since $\langle \kappa_i \rangle$ is cofinal)

Thus $f^X(i) \neq f^Y(i)$, so $f^X \neq f^Y$ $\square$



$2^{\mu_1} = 2^{\mu_0} = 2^{\kappa}$

Cofinality

Corollary

Let κ be singular infinite cardinal and assume that there are $\mu_0 < \kappa$ and λ such $2^{\mu} = \lambda$ whenever $\mu_0 \leq \mu < \kappa$. Then $2^{\kappa} = \lambda$

$$2^{\kappa} = (2^{<\kappa})^{\mathrm{cf}\,\kappa} = \lambda^{\mathrm{cf}\,\kappa} = \lambda^{\mu}$$

$$\lambda = 2^{\mu_0 \cdot \mu} = (2^{\mu_0})^{\mu} \geq \lambda^{\mu} \quad (\mu < \kappa)$$

Cofinality

Theorem (Hausdorff)

($\geq$) Trivial

For infinite cardinal κ, λ,

$$(\kappa^+)^\lambda = \kappa^\lambda \cdot \kappa^+$$

$\leq \kappa^\lambda$

($\leq$) If $\lambda \geq \kappa^+$. Then $(\kappa^+)^\lambda = 2^\lambda \leq \kappa^\lambda \cdot \kappa^+$ /

If $\lambda < \kappa^+$. Since κ^+ is regular, $(\kappa^+)^\lambda = \left| \bigcup_{\xi < \kappa^+} \xi^\lambda \right| \leq \kappa^\lambda \cdot \kappa^+$

Next on Set Theory

- Infinite sum / production
- Singular cardinal hypothesis
- Club and stationary set

Exercise

1. Define $[\kappa]^\lambda = \left\{ X \subset \kappa \;\middle|\; \operatorname{card} X = \lambda \right\}$. Show that if κ and λ are cardinals, then $\kappa^\lambda = \operatorname{card}([\kappa]^\lambda)$.
2. Show that the least κ with $\aleph_\kappa = \kappa$ is singular of cofinality ω. Show that for every regular cardinal λ there is some κ with $\aleph_\kappa = \kappa$ and $\operatorname{cf} \kappa = \lambda$