Set Theory

集合论

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Previously on Set Theory

Transfinite recursive definition It is fine to define a function $F : \mathrm{OR} \to V$ by saying that

$$F(\beta) = G(F \restriction \beta)$$

Where G is a given function.

Previously on Set Theory

Example (von Neumann hierarchy)

$$V_\alpha = \begin{cases} P(V_\beta) & \text{if } \alpha = \beta + 1 \text{ for some } \beta > \alpha \\ \bigcup_{\beta<\alpha} V_\beta & \text{otherwise} \end{cases}$$

Previously on Set Theory

Transfinite recursive definition If R is well-founded set-like relation on X. It is fine to define a function $F : X \to V$ by saying that

$$F(a) = G\Big(\big\{(b, F(b)) \bigm| bRa\big\}\Big)$$

Where G is a given function.

Previously on Set Theory

Example (Mostowski Collapse)

If E is well-founded set-like and extensional relation on X.

Then there is a collapse of (X, E) to $(M, \in)$ defined recursively on E:

$$\pi(x) = \left\{\pi(y) \,\middle|\, yEx\right\}$$

Cardinals

represents the size of sets

Cardinals

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Cardinals

f^{-1} is an α-sequence

Theorem (Well-ordering lemma)

Every set can be well-ordered, i.e. for every set A, there is a ordinal α and a bijection $f\colon A \to \alpha$, and so $A \sim \alpha$

Well-ordering Lemma:

Every set can be well-ordered

Let F be a choice function on $P(A)\setminus\{\emptyset\}$ (by AC)

Define $F^*: V \to V$: $F^*(x) = \begin{cases} F(x) & \text{if } x \in P(A)\setminus\{\emptyset\} \\ A & \text{otherwise} \end{cases}$

Define $f^*(\alpha) = F^*(A \setminus \{f(\beta) \mid \beta < \alpha\})$

Let γ be the least ordinal s.t. $f^*(\gamma) = A$ (γ exists by Rep)

Then $f = f^* \restriction \gamma$ is what we need

$A \setminus \{f(\beta) \mid \beta < \alpha\}$ $\swarrow$

$\underline{f \text{ is 1-1}}$. Assume $\beta < \alpha < \gamma$, then $f(\alpha) = F(A \setminus \{f(\beta) \mid \beta < \alpha\}) \neq f(\beta)$

$\underline{f \text{ is onto } A}$ $\quad A \setminus \operatorname{ran} f = X \neq \emptyset$

Consider $\gamma = \operatorname{dom} f$. Then $f^*(\gamma) = A$

$= F^*(A \setminus \operatorname{ran} f)$

$= F^*(X) \in X \subseteq A$ $\vdash\!\!\!\times\!\!\dashv$

Cardinals

Remarks:

- EXE: AC and well-ordering lemma are equivalent modulo ZF
- A set may not be uniquely well-ordered

Cardinals

Definition

Let x be a set. We define the cardinality of x (written $|x|$ or card x) to be the least ordinal α such that $x \sim \alpha$

Note: card is a function defined on V (AC)

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An ordinal α is a cardinal if $\alpha = \operatorname{card} \alpha$

Convention

We use typically use κ, λ, μ to denote cardinals

EXE: Every natural number is a cardinal, ω is a cardinal

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EXE: Every natural number is a cardinal, ω is a cardinal

$\omega+1$

Cardinals

Fact

- α is a cardinal iff it is the cardinality of some set x
- Some ordinals are not cardinal, and an infinite cardinal must be a limit ordinal $\alpha \sim \alpha+1$ for α infinit
- $|P(\alpha)| > \alpha$
- Cardinals are unbounded on OR (not use power set axiom)

Cardinals

Fact

- α is a cardinal iff it is the cardinality of some set x
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Cardinals

$n^+ = n+1$

Definition

$\omega^+ > \omega \cdot \omega$ $\quad \omega^{++} \ldots$

$> \omega + 1$

$> \omega + \omega$

- For each ordinal α, define α^+ to be the least cardinal $\lambda > \alpha$
- A cardinal κ is called a successor cardinal if there is some ordinal $\alpha < \kappa$ with $\kappa = \alpha^+$; otherwise we say κ is a limit cardinal

Cardinals

Assume $\alpha < \sup X$ and $\alpha \sim \sup X$
Then $\alpha < \kappa$ for some $\kappa \in X$,
and $\mathrm{card}\,\alpha < \kappa \leq \sup X$

Lemma

$\bigcup X$
$\shortparallel$

Let X be a set of cardinals, then $\sup X$ is a cardinal

Therefore, limit cardinals are unbounded (Why?)

Cardinals

Lemma

Let X be a set of cardinals, then $\sup X$ is a cardinal

Therefore, limit cardinals are unbounded (Why?)

$\sup \{\omega, \omega^{+}, \omega^{++} \ldots \} =$

$\sup \{\kappa, \kappa^{+}, \kappa^{++} \} > \kappa$

Definition

For $\alpha \in \mathrm{OR}$, we define $\aleph_\alpha$ to be the least infinite cardinal κ such that $\kappa > \aleph_\beta$ for all $\beta < \alpha$

Note: This is a transfinite recursive definition on OR

Cardinals

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Cardinals

Example

$\aleph_0 = \omega$

$\aleph_1 = \omega^+$

$\aleph_2 = \omega^{++}$

$\aleph_\omega = \sup\{\omega, \omega^+, \omega^{++}, \dots\}$

$\aleph_{\omega+1} = \aleph_\omega^+$

Cardinals

Fact

- $\alpha < \beta$ implies $\aleph_\alpha \lneq \aleph_\beta$
- $\alpha \leq \aleph_\alpha$

Lemma

Let f be an increasing function on OR. Then $\alpha \leq f(\alpha)$ for all $\alpha \in \mathrm{OR}$

- Every cardinal κ is an aleph, i.e. $\kappa = \aleph_\alpha$ for some $\alpha \in \mathrm{OR}$

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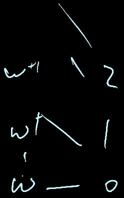


Cardinals

$\omega - \beth_\omega$

$\alpha_0 = \beth_0$

$\alpha_1 = \beth_{\alpha_0}$

$\vdots$

$\alpha_{n+1} = \beth_{\alpha_n}$

$\vdots$

$\alpha_\omega = \sup\{\beth_{\alpha_0}, \beth_{\alpha_1}, \dots\} = \beth_{\alpha_\omega}$

Fact

- $\alpha < \beta$ implies $\aleph_\alpha < \aleph_\beta$
- $\alpha \leq \aleph_\alpha$

Lemma

$\alpha < \beta \Rightarrow f(\alpha) < f(\beta)$

If $\beta < \beth_{\alpha_\omega}$. Then $\beta < \beth_\gamma$ where $\gamma < \alpha_\omega$

Let f be an increasing function on OR. Then $\alpha \leq f(\alpha)$ for all $\alpha \in \mathrm{OR}$

Then $\gamma < \alpha_n$ for some n.

- Every cardinal κ is an aleph, i.e. $\kappa = \aleph_\alpha$ for some $\alpha \in \mathrm{OR}$

Let κ be the least s.t. $\kappa \neq \aleph_\alpha$ for all $\alpha \in OR$

Then for all $\lambda < \kappa$, $\lambda = \aleph_\beta$ for some β

Let $\alpha = \sup\{\beta \mid \aleph_\beta = \lambda$, for some $\lambda < \kappa\}$, α exists, since $\aleph_\kappa \geq \kappa$

Fact

- $\alpha < \beta$ implies $\aleph_\alpha > \aleph_\beta$
- $\alpha \leq \aleph_\alpha$

Then $\kappa = \aleph_\alpha$ by definition

$> \beta$ if $\aleph_\beta = \lambda < \kappa$

Lemma

Let f be an increasing function on OR. Then $\alpha \leq f(\alpha)$ for all $\alpha \in \mathrm{OR}$

- Every cardinal κ is an aleph, i.e. $\kappa = \aleph_\alpha$ for some $\alpha \in \mathrm{OR}$

Cardinal Arithmetic

Cardinal Arithmetic

Definition

Let κ, λ be cardinal. Define

- $\kappa + \lambda = \text{card}\Big((\kappa \times \{0\}) \cup (\lambda \times \{1\})\Big)$
- $\kappa \cdot \lambda = \text{card}(\kappa \times \lambda)$
- $\kappa^{\lambda} = \text{card}\, {}^{\lambda}\kappa = \text{card}\Big(\Big\{f \,\Big|\, f : \lambda \to \kappa\Big\}\Big)$

$$2^n = |\{f \mid f : n \to \{0,1\}\}| = |P(n)|$$

Cardinal Arithmetic

EXE:

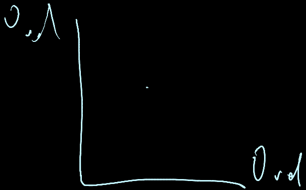
- If X, Y are disjoint sets and $\text{card}\, X = \kappa, \text{card}\, Y = \lambda$, then $\text{card}(X \uplus Y) = \kappa + \lambda$
- If $\text{card}\, X = \kappa, \text{card}\, Y = \lambda$, then $\text{card}(X \times Y) = \kappa \cdot \lambda$
- If $\text{card}\, X = \kappa, \text{card}\, Y = \lambda$, then $\text{card}(X^Y) = \kappa^\lambda$
- The usual rules hold for addition, multiplication and exponentiation

Corollary

For any cardinal κ, λ,

infinite

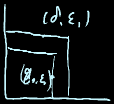
$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$$



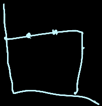
Cardinal Arithmetic

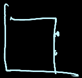
Definition (Canonical ordering on $\mathrm{OR} \times \mathrm{OR}$)

We define the canonical ordering $\leq$ on $\mathrm{OR} \times \mathrm{OR}$ as follow.We say $(\delta_0, \varepsilon_0) \leq (\delta_1, \varepsilon_1)$ if either

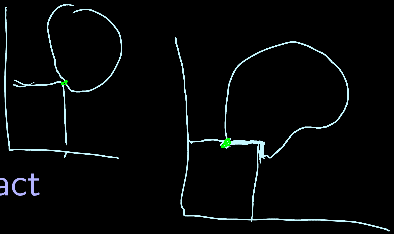


- $\max\{\delta_0, \varepsilon_0\} < \max\{\delta_1, \varepsilon_1\}$, or
- $\max\{\delta_0, \varepsilon_0\} = \max\{\delta_1, \varepsilon_1\}$, and $\delta_0 < \delta_1$, or
- $\max\{\delta_0, \varepsilon_0\} = \max\{\delta_1, \varepsilon_1\}$, $\delta_0 = \delta_1$, and $\varepsilon_0 \leq \varepsilon_1$



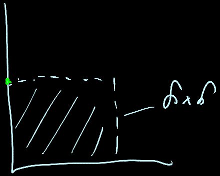


Cardinal Arithmetic



Fact

- The canonical ordering on $\mathrm{OR} \times \mathrm{OR}$ is a well-ordering and set-like
- $\delta \times \delta = \left\{ (\xi, \theta) \;\middle|\; (\xi, \theta) < (0, \delta) \right\}$

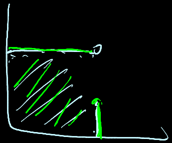


Cardinal Arithmetic

Definition (Gödel pairing function)

We define $\pi : \mathrm{OR} \times \mathrm{OR} \to \mathrm{OR}$ as follow

$$\pi(\delta, \varepsilon) = \mathrm{otp}\left\{(\xi, \theta) \;\middle|\; (\xi, \theta) < (\delta, \varepsilon)\right\}$$



Cardinal Arithmetic

Fact

$(\delta_0, \varepsilon_0) < (\delta_1, \varepsilon_1) \Rightarrow \pi(\delta_0, \varepsilon_0) < \pi(\delta_1, \varepsilon_1)$

- π is order-preserving
- π is onto OR, moreover

$$\operatorname{ran}\left(\pi \restriction \left\{(\xi, \theta) \mid (\xi, \theta) < (\delta, \varepsilon)\right\}\right) = \pi(\delta, \varepsilon)$$

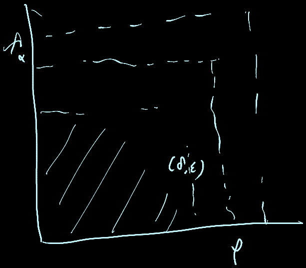
Cardinal Arithmetic

Theorem (Hessenberg)

For every α, $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$

Hessenberg Theorem:

$\aleph_\alpha \times \aleph_\alpha = \aleph_\alpha$



Proof π witness $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$

By induction,

Assume $\aleph_\alpha \cdot \aleph_\alpha > \aleph_\alpha$

Thus there is $(\delta, \varepsilon) \in \aleph_\alpha \times \aleph_\alpha$

s.t. $\pi(\delta, \varepsilon) = \aleph_\alpha$

Since $\delta, \varepsilon < \aleph_\alpha$, there ρ s.t. ($= \max\{\delta, \varepsilon\} + 1$)

$\delta, \varepsilon < \rho < \aleph_\alpha$

Consider $\rho \times \rho$, Since $\aleph_\alpha$ is cardinal, $|\rho| = \aleph_\beta$ for some $\beta < \alpha$

By IH, $|\rho \times \rho| = |\aleph_\beta \times \aleph_\beta| = \aleph_\beta \cdot \aleph_\beta = \aleph_\beta$, therefore $\aleph_\alpha \leq \aleph_\beta$ $\Rightarrow\Leftarrow$

Hessenberg Theorem:

$$\aleph_\alpha \times \aleph_\alpha = \aleph_\alpha$$

Corollary

$$\aleph_\alpha + \aleph_\alpha \le \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$$

$$\kappa + \lambda \le \kappa \cdot \lambda \le \max\{\kappa, \lambda\} \cdot \text{ma}$$

Next on Set Theory

- Cofinality
- Stationary set

Exercise

1. Prove AC from well-ordering lemma
2. Show that every natural number is a cardinal
3. Show that $\kappa + \lambda \leq \kappa \cdot \lambda$ for any cardinal $\kappa, \lambda \geq 2$