Set Theory

集合论

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Previously on Set Theory

Ordinals

$\forall y \in x \; y \subset x$

- $x \in \mathrm{OR}$ means x is transitive and well-ordered by $\underline{\in}$
- Example: $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$,..., ω

  $Sx = x \cup \{x\}$
- Ordinals are closed under successor
- $\sup X = \bigcup X = \lim X$ is an ordinal if X is a set of ordinals

  $\inf X = \bigcap X$ $\quad$ $\sup \{\omega + n \mid n \in \omega\}$
- So we have

$$\omega + 1 = S\omega, \omega + 2, \ldots, \omega + \omega = \sup_{n \in \omega}(\omega + n), \ldots$$

$n \to \omega + n$

Previously on Set Theory

Transfinite induction on OR: To show every ordinal has property P, it is sufficient to show either

- for any ordinal α, every ordinal below α has property P implies α has property P, or
- (i) $0 \in P$

  (ii) $\beta \in P \Rightarrow \beta + 1 \in P$

  (iii) $\alpha \subset P \Rightarrow \alpha \in P$ if α is limit

Previously on Set Theory

Transfinite induction on well-founded set-like relation:

Let R be well-founded set-like relation on X. To show $X \subset P$, it is sufficient to show

$$\left\{ b \mid bRa \right\} \subset P \Rightarrow a \in P$$

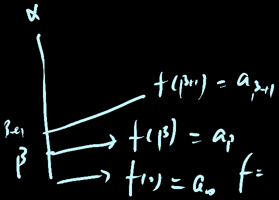
for all $a \in X$

$x + 0 = x$

$x + (y+1) = S(x+y)$

Recursive definition

Transfinite recursive definition



$f = \langle a_0, a_1, \cdots a_\beta \cdots \rangle$

Definition

Let $\alpha \leq \mathrm{OR}$. We say F (maybe class) is an α-sequence or a sequence with length α if F is a function and dom $F = \alpha$

$\varphi(x,y)$

$\forall x \forall y (\varphi(x,y) \to x \in \mathrm{OR}) \wedge \forall x \in \mathrm{OR}\ \exists! y\ \varphi(x,y)$

$x \in V$

$x = x$

Transfinite recursive definition

$x \neq y$

Theorem

$\forall x \, \exists! y \, \varphi(x, y, p)$

Let $\alpha \leq \mathrm{OR}$, $G_p : V \to V$ is a function (defined by $\varphi(x, z, p)$ with parameter p). Then there is a unique function $F_p : \alpha \to V$ (defined by some $\psi(x, z, p)$ with parameter p) such that $F_p(\beta) = G_p(F_p \restriction \beta)$ for all $\beta \in \alpha$

$\langle F(0), F(1), F(2) \ldots F(\gamma) \ldots \rangle$

$\gamma < \beta$

Transfinite recursive definition

Theorem

Let $\alpha \leq \mathrm{OR}$, $G_p : V \to V$ is a function (defined by $\varphi(x, z, p)$ with parameter p). Then there is a unique function $F_p : \alpha \to V$ (defined by some $\psi(x, z, p)$ with parameter p) such that $F_p(\beta) = G_p(F \restriction \beta)$ for all $\beta \in \alpha$

Theorem. Given $G: V \to V$, define

$F: \mathrm{OR} \to V$ such that

$$F(\alpha) = G(F \restriction \alpha) \qquad \psi(\alpha, x)$$

$(\alpha, x) \in F \iff \exists f \,[\, f$ is a function $\wedge\ f(\alpha) = x \wedge \forall \beta \in \mathrm{dom} f\ \forall \gamma < \beta\ \gamma \in \mathrm{dom} f$

$(F(\alpha) = x)$

$\wedge\ \forall \beta \in \mathrm{dom} f\ (f(\beta) = G(f \restriction \beta))\,]$

check

(1) $\forall \alpha \in \mathrm{OR}\ \exists! x\ \psi(\alpha, x)$ / (2) $F(\alpha) = G(F \restriction \alpha)$

Let α_0 be the least s.t. $\psi(\alpha_0, x_1)$, $\psi(\alpha_0, x_2)$ for some $x_1 \neq x_2$

Let f_1, f_2 witness $\psi(\alpha_0, x_1)$ and $\psi(\alpha_0, x_2)$, i.e.

$f_1(\alpha_0) = x_1$, $f_2(\alpha_0) = x_2$ and $x_1 = G(f_1 \restriction \alpha_0)$

$x_2 = G(f_2 \restriction \alpha_0)$

By Induction hypothesis: $f_1 \restriction \alpha_0 = f_2 \restriction \alpha_0$ $\qquad F(\beta) = f_1(\beta)$, $F(\beta) = f_2(\beta)$

[Assume $\exists \beta < \alpha_0$, $f_1(\beta) \neq f_2(\beta)$. But f_1, f_2 witness $\psi(\beta, \underline{f_1(\beta)})$, $\psi(\beta, \underline{f_2(\beta)})$

by IH, $f_1(\beta) = f_2(\beta)$ $\mapsto\!\leftarrow$]

Transfinite recursive definition

Corollary

Let G_1, G_2, G_3 be function on V. There is a unique function F defined on OR such that

- $F(0) = G_1(\emptyset)$
- $F(\beta + 1) = G_2(F(\beta))$
- $F(\alpha) = G_3(F \restriction \alpha)$ for limit ordinal α

Transfinite recursive definition

Definition (von Neumann universe)

Define V_α recursively for all $\alpha \in \mathrm{OR}$:

- $V_0 = \emptyset$ $= F(0)$
- $V_{\beta+1} = P(V_\beta)$ $= F(\beta+1)$ $= F(\alpha)$
- $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ for limit ordinal α

Define $WF = \bigcup_{\alpha \in \mathrm{OR}} V_\alpha$

Transfinite recursive definition

Fact

- $V = WF$
- V_α is transitive
- $\alpha < \beta \leftrightarrow V_\alpha \in V_\beta$ for any $\alpha, \beta \in \mathrm{OR}$
- $V_\alpha \cap \mathrm{OR} = \alpha$ for any $\alpha \in \mathrm{OR}$

Transfinite recursive definition

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Fact

- $V = WF$
- V_α is transitive
- $\alpha \leq \beta \leftrightarrow V_\alpha \subset V_\beta$ for any $\alpha, \beta \in \mathrm{OR}$
- $V_\alpha \cap \mathrm{OR} = \alpha$ for any $\alpha \in \mathrm{OR}$

Transfinite recursive definition

Fact

- $V = WF$
- V_α is transitive
- $\alpha < \beta \leftrightarrow V_\alpha \in V_\beta$ for any $\alpha, \beta \in \mathrm{OR}$
- $V_\alpha \cap \mathrm{OR} = \alpha$ for any $\alpha \in \mathrm{OR}$

V_α is transitive

1) $V_0 = \phi$ is transitive

2) Assume V_β is transitive, then $V_{\beta+1}$ is transitive

[$x \in V_{\beta+1} \Rightarrow x \subseteq V_\beta$, fix $y \in x$, $y \subseteq V_\beta$ since V_β is transitive, thus $y \in V_{\beta+1} = P(V_\beta)$]

3) Assume V_β is transitive for all $\beta < \alpha$

We show V_α is transitive (α is limit)

Given $x \in V_\alpha$, $x \in V_\beta$ for some β, so $x \subseteq V_\beta$

i.e. for all $y \in x$, $y \in V_\beta \subseteq V_\alpha$, so $x \subseteq V_\alpha$ $\square$

$\alpha < \beta \leftrightarrow V_\alpha \in V_\beta$

$\Rightarrow$) Ind on β.

(i) $\beta = 0$. trivial

(ii) $\beta = \gamma+1$, (a) $\alpha = \gamma$, $V_\alpha = V_\gamma \in P(V_\gamma) = V_{\gamma+1} = V_\beta$

(b) $\alpha < \gamma$, by IH, $V_\alpha \in V_\gamma \in V_{\gamma+1} = V_\beta$

since $V_{\gamma+1}$ is transitive, $V_\gamma \subseteq V_{\gamma+1}$, so $V_\alpha \in V_{\gamma+1} = V_\beta$

(iii) β is limit, $\alpha+1 < \beta$, by IH, $V_\alpha \in V_{\alpha+1} \in V_\beta$

$\Leftarrow$ Assume $V_\alpha \in V_\beta$, Since either $\alpha < \beta$, or $\alpha = \beta$, or $\beta < \alpha$

If $\alpha = \beta$, then $V_\alpha \in V_\alpha$ $\mapsto\!\leftarrow$

If $\beta < \alpha$, then $V_\alpha \in V_\beta \in V_\alpha$ $\mapsto\!\leftarrow$

$V = WF$

$V \subseteq WF$

Show by induction on $(V, \in)$

Assume $x \subseteq WF$, we show $x \in WF$

$\underbrace{\{y \in V \mid y \in x\}}_{= x} \subseteq WF$

Note $WF = \bigcup_{\alpha \in OR} V_\alpha$, i.e. $\forall y\ y \in WF \to \exists \beta (y \in V_\beta)$

Let for $y \in x$, let β_y be the least s.t. $y \in V_{\beta_y}$

Let $\alpha = \sup\{\beta_y + 1 \mid y \in x\}$, Then $x \subseteq V_\alpha$ [for any $y \in x$, $y \in V_\beta \subseteq V_\alpha$]

So $x \in P(V_\alpha) = V_{\alpha+1}$. $\square$

$V_\alpha \cap \mathrm{OR} = \alpha$

1) $V_\alpha \cap OR$ is an ordinal

i) $V_\alpha \cap OR$ is a set of ordinal, so $\in$ well-order it

ii) an intersection of transitive sets is transitive

2) Assume $V_\beta \cap OR = \beta$ for all $\beta < \alpha$

i) $\alpha = \gamma + 1$, we have $V_\gamma \cap OR = \gamma$

we show $V_{\gamma+1} \cap OR = \gamma \cup \{\gamma\}$

Assume $\theta \in V_{\gamma+1}$, then $\theta \subseteq V_\gamma$, so. $\theta \subseteq \gamma = V_\gamma \cap OR$

$\theta \subseteq \gamma \in \gamma+1$

$\theta \in \gamma+1$, i.e. $\theta \leq \gamma$, if $\theta = \gamma$, then, $[\gamma \subseteq V_\gamma]$

$\theta \in V_{\gamma+1} \cap OR$

if $\theta < \gamma$, ...

ii) α is limit, i.e. $\alpha = \bigcup \alpha$

$= \bigcup \{\beta \mid \beta < \alpha\} = \bigcup \{V_\beta \cap OR \mid \beta < \alpha\} = V_\alpha \cap OR$

Transfinite recursive definition

Definition (rank)

For each set $x \in V = WF$, define $\operatorname{rank} x$ to be the least ordinal α such that $x \in V_{\alpha+1}$

Fact

If $x \in y$, then $\operatorname{rank} x < \operatorname{rank} y$

IND on y

Transfinite recursive definition

Theorem (Transfinite recursive definition on V)

$G_p : V \to V$ be a function. Then there is a unique function $F_p : V \to V$ such that

$$F_p(a) = G_p(F_p{\restriction}a) = G_p(\{(b, F_p(b)) \mid b \in a\})$$ $F_\alpha : V_{\alpha+1} \to V$

$H(\alpha) = F_\alpha$ s.t. $F_\alpha(x) = G(F_\alpha{\restriction}x)$

for any set a for all x s.t. $\operatorname{rank} x < \alpha$

$F = \bigcup_{\alpha \in On} F_\alpha$

Transfinite recursive definition

Theorem (Transfinite recursive definition on well-founded relation)

Let R be a well-founded set-like relation on X, $G_p : V \to V$ is a function. There is then a function $F_p : X \to V$ such that

$$F_p(a) = G_p\left(\left\{(b, F_p(b)) \;\middle|\; bRa\right\}\right)$$

for any $a \in X$

Proof:

$$F(x) = a \iff \exists f\, \big[f(x) = a \wedge \forall y \in \operatorname{dom} f\ \forall z R y\ z \in \operatorname{dom} f$$
$$\wedge\, \forall y\, f(y) = G(f \restriction \{z \in X \mid z R y\}) \big]$$

Mostowski Collapse

Definition

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A binary relation E on X is extensional if for all $x, y \in \cancel{M}\,X$

$$x = y \Leftrightarrow (\forall z \in X)(zEx \leftrightarrow zEy)$$

Fact

$\in$ is extensional on any transitive class

Mostowski Collapse

Theorem (Mostowski Collapse)

Let E be a well-founded, set-like and extensional relation on X. There is a transitive class M and a bijection $\pi : \cancel{E}\,X \to M$ such that for any $x, y \in X$

$$xEy \Leftrightarrow \pi(x) \in \pi(y)$$

M. collapse

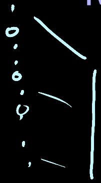
We call $\pi/(M, \in)/M$ the ~~transitive closure~~ of $(X, E)/E/X$

Mostowski collapse theorem

$$\pi(x) = \underbrace{\{\pi(y) \mid y \bar{E} x\}}_{\shortparallel}$$

$$G(\langle \pi(y) \mid y \bar{E} x \rangle)$$

Mostowski Collapse



Example

Let $X \subset \mathrm{OR}$. Then $\in$ is well-founded (ordered), set-like and extensional on X. The Mostowski collapse $\pi : X \to \alpha$ of Let X is unique and α is an ordinal. We define the order type of X (written $\operatorname{otp} X$) to be this α.

Transfinite recursive definition

Lemma

For each set x there is a "smallest" transitive set y such that $x \in y$

Definition (transitive closure)

For each set x define the y in the above lemma to be the **transitive closure** of x, written $\mathrm{TC}\{x\}$

Next on Set Theory

- Cardinals
- Simple facts on cardinal arithmetic

Exercise

1. Given any formula $\varphi(x)/\psi(x,y)$, prove in ZFC:

$$\mathrm{Fund}_\varphi:\ \exists x\varphi(x) \to \exists x(\varphi(x) \wedge \forall y \in x \neg\varphi(y))$$

$$\mathrm{Coll}_\psi:\ \forall x \exists y \psi \to \forall A \exists B \forall x \in A \exists y \in B \psi$$

Exercise

2. Let $(W_0, \leq_0), (W_1, \leq_1)$ be well-ordering.

   Define $\leq = \leq_0 \oplus \leq_1$ to be the relation on $W_0 \times \{0\} \cup W_1 \times \{1\}$ such that $(x, i) \leq (y, j)$ iff $i < j$, or $i = j$ and $x \leq_i y$.

   Define $\leq = \leq_0 \otimes \leq_1$ to be the relation on $W_0 \times W_1$ such that $(x_0, y_0) \leq (x_1, y_1)$ iff $x_0 <_0 x_1$, or $x_0 = x_1$ and $y_0 \leq_1 y_1$

   Ordinal arithmetic: Use recursion theorem to define $\alpha + \beta$, $\alpha \cdot \beta$ such that $\alpha + \beta = \mathrm{otp}(\alpha \oplus \beta)$ and $\alpha \cdot \beta = \mathrm{otp}(\alpha \otimes \beta)$.

   Define α^β in the same way, and $(*)$ try to make sense of it.

Exercise

3. Show that $+$ and $\cdot$ are associative but not commutative
4. Ordinal division: Given $\alpha, \gamma \in \mathrm{OR}$ and $\alpha > 0$. Show that there exists a unique β and unique ρ such that
$$\gamma = \alpha \times \beta + \rho$$
5. Cantor normal form theorem: Given $\alpha > 0$. Show that there are unique positive natural numbers k and $c_1, \ldots c_k$ and ordinals $0 \leq \beta_1 < \ldots < \beta_k$ such that
$$\alpha = \omega^{\beta_k} \cdot c_k + \cdots + \omega^{\beta_1} \cdot c_1$$