Set Theory

集合论

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Previously on Set Theory

- Partial order: reflexivity, antisymmetry, transitivity
- Linear order: total partial order

  极大元/极小元
- maximal/minimal element

  最大元/最小元
- maximum/minimum
- upper bound/lower bound
- supremum/infimum

Previously on Set Theory

Well-ordering

- Definition: Every nonempty subset has a minimum
- Facts:
  - Well-ordering implies total $f: W \to W$
  - order-preserving implies nondecreasing
  - isomorphism is unique $f: W_1 \simeq W_2$
  - For any pair of well-orderings, either the two are isomorphic, or one is isomorphic to a proper initial segment of the other.

Ordinals

representations of well-orderings

Ordinals

representations of well-orderings

Ordinals

Definition

$x \in X,\ y \in x \Rightarrow y \in X$

- A set X is said to be transitive if for every $x \in X$, $x \subset X$
- A set α is called an ordinal if α is transitive and total ordered by $\underline{\in}$ $\Rightarrow$ well-ordering (by foundation)

Let $x \in \mathrm{OR}$ be abbr. for x is an ordinal

Ordinals

$\{0, 2\}$

Example

- $\emptyset$ $= 0$
- $\{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$

  $= 1 = \{0\}$ $= \{0, 1\} = 2$

  $3 = \{0, 1, 2\}$
- ω

  $= \{0, S0, SS0, \ldots\}$

  $S0 = 0 \cup \{0\} = \{0\} = 1$

Ordinals

Lemma

1. If α is an ordinal, then $S\alpha$ is an ordinal $=\alpha+1$
2. If α is an ordianl and $\beta \in \alpha$, then β is an ordinal
3. If α, β are ordinals, and $\alpha \subsetneq \beta$, then $\alpha \in \beta$
4. If α, β are ordinals, then either $\alpha \overset{<}{\in} \beta$, or $\alpha = \beta$, or $\beta \overset{<}{\in} \alpha$

1 $\alpha \in \mathrm{OR} \Rightarrow S\alpha \in \mathrm{OR}$

Proof Note: $S\alpha = \alpha \cup \{\alpha\}$

(1) $S\alpha$ is transitive

Let $\beta \in \alpha \cup \{\alpha\}$. Then either $\beta \in \alpha$, or $\beta = \alpha$

Assume $\beta \in \alpha$,

Since α is transitive, $\beta \subseteq \alpha \subseteq \alpha \cup \{\alpha\}$

Assume $\beta = \alpha$, Trivial

(2) To check $\alpha \cup \{\alpha\}$ is linearly ordered by $\in$;

(a): Given $\gamma_1, \gamma_2, \gamma_3$. Assume $\gamma_1 \in \gamma_2$, $\gamma_2 \in \gamma_3$

If $\gamma_3 \in \alpha$, then $\gamma_2, \gamma_1 \in \alpha$. [$\alpha$ is transitive]

Since $\in$ is transitive on α, we have $\gamma_1 \in \gamma_3$

Otherwise $\gamma_3 = \alpha$

Again, by transitivity of α,

$\gamma_1 \in \alpha = \gamma_3$

2 $\alpha \in \mathrm{OR}, \beta \in \alpha \Rightarrow \beta \in \mathrm{OR}$

Proof. Assume $\beta \in \alpha \in \mathrm{OR}$

(1) β is transitive.

Fix $\gamma \in \beta \subseteq \alpha$, and $\theta \in \gamma$

We want to show $\theta \in \beta$

We know $\theta \in \gamma \subseteq \alpha$

Since α is totally ordered by $\in$

either $\theta \in \beta$, or $\theta = \beta$, or $\beta \in \theta$

if $\theta = \beta$, then

$$\underset{\theta}{\underset{\|}{\beta}} \ni \gamma \ni \beta \ni \gamma \ni \beta \cdots$$

else if $\beta \in \theta$

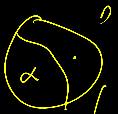
$$\beta \ni \gamma \ni \theta \ni \beta \ni \gamma \ni \cdots$$

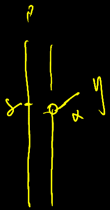
Therefore $\theta \in \beta$

(2) β is total ordered by $\in$.

Since $\beta \subseteq \alpha$ is a suborder

3 $\alpha,\beta \in \mathrm{OR}, \alpha \subsetneq \beta \Rightarrow \alpha \in \beta$





Proof

Let $\gamma = \min \beta \setminus \alpha$

Claim $\gamma = \alpha$

Fix $\theta \in \gamma$,

since γ is the minimal

$\theta \notin \beta \setminus \alpha$

Since β is transitive, $\theta \in \beta$,

thus $\theta \in \alpha$, so $\gamma \subseteq \alpha$

Fix $\eta \in \alpha$,

if $\eta \notin \gamma$, then $\eta = \gamma$ or $\gamma \in \eta$

First $\eta \neq \gamma = \min \beta \setminus \alpha \notin \alpha$

Second, if $\gamma \in \eta$, and,

$\eta \in \alpha$, but $\gamma \notin \alpha$,

but α is transitive

4 $\alpha, \beta \in \mathrm{OR}$

$\Rightarrow \alpha \in \beta$, or $\alpha = \beta$, or $\beta \in \alpha$

Proof

It is sufficient to show either $\alpha \subseteq \beta$, or $\beta \subseteq \alpha$ [by (3).]

Let $\gamma = \alpha \cap \beta$

Then γ is an ordinal

Claim $\gamma = \alpha$ or $\gamma = \beta$

Note: $\gamma \subseteq \alpha$, and $\gamma \subseteq \beta$.

if $\gamma \neq \alpha$, then $\gamma \in \alpha$

if $\gamma \neq \beta$ .. $\gamma \in \beta$

If $\gamma \neq \alpha$ and $\gamma \neq \beta$, then

$\gamma \in \alpha \cap \beta = \gamma$ $\not\vdash$

Ordinals

X

$= \cap X$

Lemma

- Let X be a nonempty set of ordinals, then $\bigcap X$ is the minimum of X
- Let X be a set of ordinals, then $\bigcup X$ is also an ordinal

$\cup X$

Notation

In this case, we usually write $\inf X$ / $\sup X$ instead of $\bigcap X / \bigcup X$

Ordinals

Lemma

- Let X be a nonempty set of ordinals, then $\bigcap X$ is the minimum of X
- Let X be a set of ordinals, then $\bigcup X$ is also an ordinal

$\bigcup \omega = \omega$

Notation

In this case, we usually write $\inf X$ / $\sup X$ instead of $\bigcap X / \bigcup X$

Ordinals

Assume OR is set, then OR is an ordinal

i.e. $\mathrm{OR} \in \mathrm{OR}$ Help

Fact

OR is not a set

Convention

Usually, we call things like OR a (proper) class

Ordinals

Fact

OR is not a set

Convention

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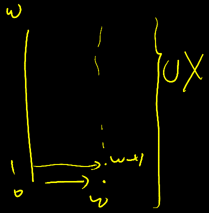
Usually, we call things like OR a (proper) class

Ordinals

$\omega + \omega$



Definition

Let α be an ordinal. We say

- α is a successor ordinal, if there is some ordinal β such that $\alpha = S\beta = \beta + 1$
- α is a limit ordinal, if it is not a successor ordinal.

e.g. ω

$\underline{x \in C}$ $\underline{\varphi(x, p)}$ parameter

Classes

In a 1st-order set theory, e.g. ZFC, when we talk about a class, we are actually talking with a 1st-order set theory formula.

For example, $x \in V$ is an abbr. for some $x = x$. Let C be a class. We can always say

- C is a "relation" $\forall x \, (\varphi(x, p) \to x \text{ is a } \underset{\exists y \exists z \; x=(y,z)}{\underline{\text{ordered pair}}})$
- C is a "function" $\forall y \, \exists! z \; \varphi((y,z), p))$
- The domain/range of C is some other class D

...... $\underline{D = \operatorname{dom} C}$ $\forall x \, (\psi(x) \leftrightarrow \exists y \; \varphi(x, y))$

$\psi(x)$ $\varphi(x,y)$

Definition

$(V, \subseteq)$

Let $(\mathbb{P}, \leq)$ be a class partial-ordering, we say $\leq$ is set-like, if for each $p \in \mathbb{P}$, $\left\{q \in P \,\middle|\, q \leq p\right\}$ is a set.

Lemma

Let $(W, \leq)$ be a set-like class well-ordering. That is every nonempty subset of P has a $\leq$-minimal element. Then every nonempty subclass of P has a $\leq$-minimal element.

$C \subseteq W$, since $C \neq \emptyset$, Let $p \in C$.

Then $\{q \mid q \in C \wedge q \leq p\}$ is a subset of W

Definition

Let q be a minimal-element of $\{q \mid q \leq p\}$

Let $(\mathbb{P}, \leq)$ be a class partial-ordering, we say $\leq$ is set-like, if for each $p \in \mathbb{P}$, $\left\{q \in P \,\middle|\, q \leq p\right\}$ is a set.

Lemma

Let $(W, \leq)$ be a set-like class ~~well~~ Partial-ordering. That is every nonempty subset of P has a $\leq$-minimal element. Then every nonempty subclass of P has a $\leq$-minimal element.

Classes

$\mathbb{P} = \{x \mid \varphi_{\mathbb{P}}(x)\}$

Definition

Let $(\mathbb{P}, \leq)$ be a class partial-ordering, we say $\leq$ is set-like, if for each $p \in \mathbb{P}$, $\left\{q \in P \,\middle|\, q \leq p\right\}$ is a set.

Lemma (Schema) $\varphi_{(W,\leq)}(x)$ $\varphi_{P}(x)$

Let $(W, \leq)$ be a set-like class well-ordering. That is every nonempty subset of P has a $\leq$-minimal element. Then every nonempty subclass of P has a $\leq$-minimal element.

$\exists y(\varphi_P(y) \wedge \forall z(\varphi_P(z) \to \neg\varphi_{\leq}(z,y)))$

Transfinite induction

Transfinite induction

Recall:

Lemma

Let $(W, \leq)$ be a well-ordering, and $P \subset W$. Assume that for each $x \in W$, $\forall y < x \; y \in P$ implies $x \in P$. Then $P = W$

Transfinite induction

Lemma (Transfinite induction)

Let $P \subset \mathrm{OR}$, and for each $\alpha \in \mathrm{OR}$, $\alpha \subset P$ implies $\alpha \in P$. Then $P = \mathrm{OR}$

Transfinite induction

ω

Lemma

Let $P \subset \mathrm{OR}$. Assume

- $0 \in P$
- For any $\alpha \in \mathrm{OR}$, $\alpha \in P$ implies $\alpha + 1 \in P$
- For any limit ordinal α, if there is $S \subset P$ such that $\sup S = \alpha$, then $\alpha \in P$

Then $P = \mathrm{OR}$

Transfinite induction

$\subseteq$

Definition

We say a relation $R \subset X^2$ (maybe a class) is well-founded if every nonempty subset $x \subset X$ has an R-minimal element. Otherwise, we say R is ill-founded.

Transfinite induction

Theorem (Induction on well-founded relation)

Let $R \subset X^2$ be well-founded and set-like. Let $Y \subset X$ be such that for all $a \in X$, $\left\{b \,\middle|\, bRa\right\} \subset Y$ implies $a \in Y$. Then $X = Y$

Transfinite induction

von Neumann universe

Definition

Define V_α recursively for all $\alpha \in \mathrm{OR}$:

- $V_0 = \emptyset$
- $V_{\beta+1} = P(V_\beta)$
- $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ for limit ordinal α

Define $WF = \bigcup_{\alpha\in\mathrm{OR}} V_\alpha$

$$\frac{x \in WF}{\varphi_{=2}(x)} \iff x = y$$

Next on Set Theory

- Transfinite recursion
- Mostowski collapse

Exercise

1. Which of the following is true? If true, prove it, otherwise, give a counterexample.
   - If X, Y are transitive, then $X \cup Y$ is transitive
   - If X, Y are transitive, then $X \cap Y$ is transitive
   - If X is transitive and $Y \in X$, then Y is transitive
   - If X are transitive and $Y \subset X$, then Y is transitive
   - If X is transitive and $S \subset P(X)$, then $X \cup S$ is transitive

Exercise

2. Show that a relation $R \subset X^2$ is well-founded if and only if there is no infinite sequence

$$\ldots x_2 R x_1 R x_0$$

3. Show that V_α is transitive for all $\alpha \in \mathrm{OR}$
4. Which axioms of ZFC holds in $(V_\alpha, \in \restriction V_\alpha)$ where α is a limit ordinal?