Set Theory

集合论

杨睿之

yangruizhi@fudan.edu.cn

School of Philosophy, Fudan University

Fall 2015

Previously on Set Theory

ZFC Axioms

- Axiom of extensionality
- Axiom of foundation
- Axiom of pairs
- Axiom of Union

Previously on Set Theory

- Axiom of power set
- Separation schema
- Axiom of infinity
- Replacement Schema
- Axiom of choice

Previously on Set Theory

Defined notions

Previously on Set Theory

- Relation

  domain, range, inverse, image

- Function

  injective, surjective, bijective, composition, restriction, B^A

Order

The order of the size/rank/... of sets

Partial order

Definition

We say $\leq$ is a partial order on $\mathbb{P}$ if $\leq$ be a binary relation on $\mathbb{P}$ ($\leq \subset \mathbb{P} \times \mathbb{P}$), and the following hold

- $\forall x \in \mathbb{P}(x \leq x)$ (reflexivity)
- $\forall x, y \in P(x \leq y \rightarrow y \leq x \rightarrow x = y)$ (antisymmetry)
- $\forall x, y, z \in \mathbb{P}(x \leq y \rightarrow y \leq z \rightarrow x \leq z)$ (transitivity)

We say $(\mathbb{P}, \leq)$ (or simply $\mathbb{P}$) is a partial order if $\leq$ is a partial order on $\mathbb{P}$.

Partial order

Definition

We say $\leq$ is a partial order on $\mathbb{P}$ if $\leq$ be a binary relation on $\mathbb{P}$ ($\leq \subset \mathbb{P} \times \mathbb{P}$), and the following hold

- $\forall x \in \mathbb{P}(x \leq x)$ (reflexivity)
- $\forall x, y \in P(x \leq y \to y \leq x \to x = y)$ (antisymmetry)
- $\forall x, y, z \in \mathbb{P}(x \leq y \to y \leq z \to x \leq z)$ (transitivity)

We say $(\mathbb{P}, \leq)$ (or simply $\mathbb{P}$) is a partial order if $\leq$ is a partial order on $\mathbb{P}$.

Partial order

Convention

Let $(\mathbb{P}, \leq)$ be a partial order. By $x < y$, we mean $x \leq y \wedge x \neq y$

Fact

Let $\leq$ be a partial order on $\mathbb{P}$, $X \subset \mathbb{P}$. Then $\leq \cap X^2$ is a partial order on X.

In this case, we say $(X, \leq)$ (abbr. for $(X, \leq \cap X^2)$) is a suborder of $(\mathbb{P}, \leq)$

Partial order

Example

- The natural ordering of natural numbers $(\mathbb{N}, \leq)$ is a partial order
- EXE: $\subset$ is a partial order on $P(A)$ for any set A
- The ordering of authority

Partial order

Definition

A partial order $(L, \leq)$ is linear (or total) if

$$\forall x, y \in L (x \leq y \vee y \leq x)$$

Example

- the natural ordering of natural numbers
- the ordering of value vs the ordering of price
-

Partial order

Definition

Let $(\mathbb{P}, \leq)$ be a partial order. We say

- $x \in \mathbb{P}$ is a maximal element of $\mathbb{P}$ if

$$\forall y \in \mathbb{P} x \nless y$$

- $x \in \mathbb{P}$ is the maximum of $\mathbb{P}$ if

$$\forall y \in \mathbb{P} y \leq x$$

EXE: maximum/minimum of a partial order is unique if exists

Partial order

Definition

Let $(\mathbb{P}, \leq)$ be a partial order. We say

- $x \in \mathbb{P}$ is a maximal element of $\mathbb{P}$ if

$$\forall y \in \mathbb{P} x \not< y$$

- $x \in \mathbb{P}$ is the maximum of $\mathbb{P}$ if

$$\forall y \in \mathbb{P} y \leq x$$

EXE: maximum/minimum of a partial order is unique if exists

Partial order

Definition

Let $(\mathbb{P}, \leq)$ be a partial order. We say

- $x \in \mathbb{P}$ is a minimal element of $\mathbb{P}$ if

$$\forall y \in \mathbb{P}\, y \nless x$$

- $x \in \mathbb{P}$ is the maximum of $\mathbb{P}$ if

$$\forall y \in \mathbb{P}\, y \leq x$$

EXE: maximum/minimum of a partial order is unique if exists

Partial order

Definition

Let $(\mathbb{P}, \leq)$ be a partial order. We say

- $x \in \mathbb{P}$ is a minimal element of $\mathbb{P}$ if

$$\forall y \in \mathbb{P} y \not< x$$

- $x \in \mathbb{P}$ is the maximum of $\mathbb{P}$ if

$$\forall y \in \mathbb{P} y \leq x$$

EXE: maximum/minimum of a partial order is unique if exists

Partial order

Definition

Let $(\mathbb{P}, \leq)$ be a partial order. We say

- $x \in \mathbb{P}$ is a minimal element of $\mathbb{P}$ if
$$\forall y \in \mathbb{P} y \nless x$$

- $x \in \mathbb{P}$ is the maximum of $\mathbb{P}$ if
$$\forall y \in \mathbb{P} y \leq x$$

EXE: maximum/minimum of a partial order is unique if exists

Partial order

Definition

Let $(\mathbb{P}, \leq)$ be a partial order. We say

- $x \in \mathbb{P}$ is a minimal element of $\mathbb{P}$ if

$$\forall y \in \mathbb{P} y \not< x$$

- $x \in \mathbb{P}$ is the minimum of $\mathbb{P}$ if

$$\forall y \in \mathbb{P} x \leq y$$

EXE: maximum/minimum of a partial order is unique if exists

Partial order

Definition

Let $(\mathbb{P}, \leq)$ be a partial order, $A \subset \mathbb{P}$. We say

- $x \in \mathbb{P}$ is an upper bound of A if

$$\forall y \in A y \leq x$$

- x is the supremum of A if

$$x \text{ is the minimum of the set of all upper bound of } A$$

Partial order

Definition

Let $(\mathbb{P}, \leq)$ be a partial order, $A \subset \mathbb{P}$. We say

- $x \in \mathbb{P}$ is an upper bound of A if

$$\forall y \in A y \leq x$$

- x is the supremum of A if

$$x \text{ is the minimum of the set of all upper bound of } A$$

Partial order

Definition

Let $(\mathbb{P}, \leq)$ be a partial order, $A \subset \mathbb{P}$. We say

- $x \in \mathbb{P}$ is an lower bound of A if

$$\forall y \in A x \leq y$$

- x is the supremum of A if

$$x \text{ is the minimum of the set of all upper bound of } A$$

Partial order

Definition

Let $(\mathbb{P}, \leq)$ be a partial order, $A \subset \mathbb{P}$. We say

- $x \in \mathbb{P}$ is an lower bound of A if

$$\forall y \in A x \leq y$$

- x is the supremum of A if

  x is the minimum of the set of all upper bound of A

Partial order

Definition

Let $(\mathbb{P}, \leq)$ be a partial order, $A \subset \mathbb{P}$. We say

- $x \in \mathbb{P}$ is an lower bound of A if

$$\forall y \in A x \leq y$$

- x is the supremum of A if

  x is the minimum of the set of all upper bound of A

Partial order

Definition

Let $(\mathbb{P}, \leq)$ be a partial order, $A \subset \mathbb{P}$. We say

- $x \in \mathbb{P}$ is an lower bound of A if

$$\forall y \in A x \leq y$$

- x is the infimum of A if

  x is the maximum of the set of all lower bound of A

Partial order

Notation

Let $(\mathbb{P}, \leq)$ be a partial order, and $X \subset \mathbb{P}$. We write

- $\max X =$ the maximum of $(X, \leq)$

  $\min X =$ the minimal of X

- $\sup X =$ the supremum of X

  $\inf X =$ the infimum of X

if they exists

Partial order

Definition

Let $(\mathbb{P}_1, \leq_1), (\mathbb{P}_2, \leq_2)$ be partial orders. A function $f : \mathbb{P}_1 \to \mathbb{P}_2$ is said to be order-preserving if for all $x, y \in \mathbb{P}_1$,

$$x <_1 y \Leftrightarrow f(x) <_2 f(y)$$

If $f : \mathbb{P}_1 \to \mathbb{P}_2$ is also a surjection, we say f is an isomorphism from $\mathbb{P}_1$ to $\mathbb{P}_2$ (written $f : \mathbb{P}_1 \cong \mathbb{P}_2$). In this case, we also say $\mathbb{P}_1$ and $\mathbb{P}_2$ are isomorphic (written $\mathbb{P}_1 \cong \mathbb{P}_2$).

Well-ordering

Well-ordering

Definition

We say partial order $(W, \leq)$ is a well-ordering if for every nonempty $A \subset W$, $\min A$ exists

Well-ordering is good

Well-ordering

Fact

Well-ordering implies total

Well-ordering

Lemma

- Let $(W, \leq)$ be a well-ordering, and $f : W \to W$ is order-preserving. Then f is nondecreasing, i.e. $f(x) \geq x$ for all $x \in W$. Furthermore, if $f : W \cong W$, then f is the identity function on W
- The function witnesses $(W_1, \leq) \cong (W_2, \leq)$ is unique

Well-ordering

Definition

Let $(W, \leq)$ be a well-ordering and $x \in W$. An initial segment of W given by x is defined to be

$$W{\upharpoonright}x = \left\{ z \in W \,\middle|\, z < y \right\}$$

Lemma

Let W be a well-ordering and $x \in W$. Then $W \not\cong W{\upharpoonright}x$

Well-ordering

Induction on well-ordering

Lemma

Let $(W, \leq)$ be a well-ordering, and $P \subset W$. Assume that for each $x \in W$, $\forall y < x \; y \in P$ implies $x \in P$. Then $P = W$

A proof of universality can be turned into a proof of some local cases.

Well-ordering

Induction on well-ordering

Lemma

Let $(W, \leq)$ be a well-ordering, and $P \subset W$. Assume that for each $x \in W$, $\forall y < x \; y \in P$ implies $x \in P$. Then $P = W$

A proof of universality can be turned into a proof of some local cases.

Well-ordering

Theorem

Let $(W_1, \leq_1)$ and $(W_2, \leq_2)$ be well-ordering. Then exactly one of the following statements holds

1. $W_1 \cong W_2$
2. $\exists y \in W_2 \; W_1 \cong W_2 \restriction y$
3. $\exists x \in W_1 \; W_1 \restriction x \cong W_2$

Theorem. exactly one of the following holds

1. $W_1 \cong W_2$
2. $\exists y \in W_2 \ W_1 \cong W_2 \upharpoonright y$
3. $\exists x \in W_1 \ W_1 \upharpoonright x \cong W_2$

Next on Set Theory

- Oridnals
- Transfinite Induction and Recursion
- Von Neumann universe