Set Theory

集合论

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Fall 2015

Previously on Set Theory

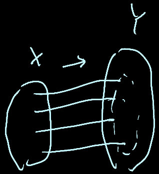
助教：

- 单芃舒（14210160029@fudan.edu.cn）
- 微信互助小组：集合论 2015 秋

习题、考试

- 自愿提交，习题课（正课课后）
- 开卷考试

Previously on Set Theory



- What set theory is about?
- How big is a set?
    - Cantor's definition of $\leq$ and $\sim$
    - Cantor Theorem: there is another infinity!
    - Cantor-Bernstein Theorem $X \leq Y, Y \leq X \rightarrow X \sim Y$

Set theory is about the sizes of sets (of reals).

Previously on Set Theory

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  - Cantor's definition of $\leq$ and $\sim$
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Set theory is about the sizes of sets (of reals).

Previously on Set Theory

- But what is a size/number? Caesar problem
- Frege's definition of number
- Frege's assumption and Russell's Paradox

Here comes the axiomatic set theory, ZFC

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Here comes the axiomatic set theory, ZFC

$$\exists z (z \in X)$$

Previously on Set Theory

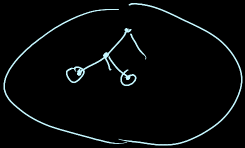
$$\forall X (X \neq \emptyset \rightarrow \exists y (y \in X \wedge y \cap X = \emptyset))$$

$$\neg \exists z (z \in y \wedge z \in X)$$

ZFC:

- Formal language
- Axiom of extensionality
- Axiom of foundation
- Axiom of pairs

$$\forall X \forall Y (X = Y \leftrightarrow \forall z (z \in X \leftrightarrow z \in Y))$$



$$\{x, y\} = z$$

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w = x \vee w = y)$$

Zermelo-Fraenkel Set Theory

Definition (Singleton)

We write $y \in \{x\}$ / $Y = \{x\}$ instead of $x \in \{y, y\}$ / $X = \{y, y\}$

Fact

$\forall x \exists Y \forall Y' (Y' = Y \leftrightarrow Y' = \{x\})$

Zermelo-Fraenkel Set Theory

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Fact

$\forall X \exists! Y\ Y\{x\}$

Zermelo-Fraenkel Set Theory

$X = \{Y_0, Y_1, Y_2\}$

$z \in \bigcup X \leftrightarrow z \in Y_0 \vee z \in Y_1 \vee z \in Y_2$

Axiom of union

$$\forall X \exists Y y = \bigcup X$$

where $z \in \bigcup X$ is abbr. for $\exists Y (Y \in X \wedge z \in Y)$

Fact (Pair + Union)

$\forall X \forall Y \exists Z (Z = X \cup Y)$, where $X \cup Y = \bigcup \{X, Y\}$

Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

Axiom of power set

$\forall w(w \in z \to w \in X)$

$$\forall X \exists Y \forall Z (Z \in Y \leftrightarrow Z \subset X)$$

Let $Z \in P(X)$ be an abbr. for $Z \subset X$, and $Y = P(X)$ for $\forall Z (Z \in Y \leftrightarrow Z \subset X)$

Zermelo-Fraenkel Set Theory

Axiom of power set

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Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

What kind of set we have now?

We know we have some, but we don't know what we have yet

Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

$x \neq x$

Separation schema: For each formula of the language of set theory $\varphi(x, v_1, \ldots, v_k)$, the following is an axiom of separation

$$\forall A \exists B \forall x (x \in B \leftrightarrow x \in A \wedge \varphi(x, v_1, \ldots, v_k))$$

Let $B = \{x \in A \mid \varphi(x)\}$ be abbr. for $\forall x (x \in B \leftrightarrow x \in A \wedge \varphi(x))$

Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

Example

$\{x \in Y \mid x \neq x\} = \emptyset$

$R = \{x \in A \mid x \notin x\}$

$R \in R$?

- Empty set exists: $\exists x\ x = \emptyset$
- Intersection exists I: $\forall X \forall Y \exists Z\ Z = X \cap Y$

  where $Z = X \cap Y$ is abbr. for $\forall x (x \in Z \leftrightarrow z \in X \land x \in Y)$
- Intersection exists II: $\forall A (A \neq \emptyset \to \exists B\ B = \bigcap A)$

  where $B = \bigcap A$ is abbr. for

  $\forall x (x \in B \leftrightarrow \forall y (y \in A \to x \in y))$

Zermelo-Fraenkel Set Theory

Example

- Empty set exists: $\exists x\ x = \emptyset$
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  where $B = \bigcap A$ is abbr. for

  $\forall x(x \in B \leftrightarrow \forall y(y \in A \rightarrow x \in y)$

Zermelo-Fraenkel Set Theory

$x \in A$

Example

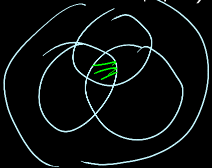
$\{z \in X \mid \forall Y (Y \in A \to z \in Y)\}$

- Empty set exists: $\exists x \; x = \emptyset$
- Intersection exists I: $\forall X \forall Y \exists Z \; Z = X \cap Y$ $\{z \in X \mid z \in Y\}$

  where $Z = X \cap Y$ is abbr. for $\forall x (x \in Z \leftrightarrow z \in X \wedge x \in Y)$
- Intersection exists II: $\forall A (A \neq \emptyset \to \exists B \; B = \bigcap A)$

  where $B = \bigcap A$ is abbr. for

  $\forall x (x \in B \leftrightarrow \forall y (y \in A \to x \in y))$



A

Zermelo-Fraenkel Set Theory

Example

- $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\} \ldots$
- Define Sx (the successor of x) to be the set $x \cup \{x\}$

  EXE: Every set has a successor

  $\emptyset, S\emptyset = \{\emptyset\}, SS\emptyset = \{\emptyset, \{\emptyset\}\}, \ldots, S^n\emptyset, \ldots$
- $\emptyset, P(\emptyset), P(P(\emptyset)), \ldots, P^n(\emptyset) \ldots$

Zermelo-Fraenkel Set Theory

Example

$S\phi = \phi \cup \{\phi\} = \{\phi\}$

$S\{\phi\} = \{\phi\} \cup \{\{\phi\}\} = \{\phi, \{\phi\}\}$

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- $\emptyset, P(\emptyset), P(P(\emptyset)), \dots, P^n(\emptyset) \dots$

  $P(\emptyset) = \{\emptyset\}$, $P(P(\emptyset)) = \{\emptyset, \{\emptyset\}\}$

Zermelo-Fraenkel Set Theory

Definition

When we say X is inductive, we mean

$$\emptyset \in X \wedge \forall y (y \in X \rightarrow Sy \in X)$$

Axiom of infinity

$$\exists X \; X \text{ is inductive}$$

Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

Definition

Define ω to be the "smallest inductive set", i.e. $x = \omega$ is abbr. for

$$x \text{ is inductive } \wedge \forall y (y \text{ is inductive } \rightarrow x \subset y)$$

Fact

ω is well-defined (unique) and exists.

Zermelo-Fraenkel Set Theory

Definition

Define ω to be the "smallest inductive set", i.e. $x = \omega$ is abbr.

for

$$x \text{ is inductive } \wedge \forall y (y \text{ is inductive } \rightarrow x \subset y)$$

By Inf, there is an inductive set Y

Let $\omega = \{x \in Y \mid \forall X (X \text{ is inductive} \rightarrow x \in X)\}$

$\emptyset \in \omega$, if $z \in \omega$ ... $Sz \in \omega$

Fact

ω is well-defined (unique) and exists.

Assume A is IND, then $\omega \subseteq A$

Therefore ω is IND, Assume ω' is "smallest IND", then $\omega \subseteq \omega'$, and $\omega' \subseteq \omega$

Z

Zermelo-Fraenkel Set Theory

ZF

Zermelo-Fraenkel Set Theory

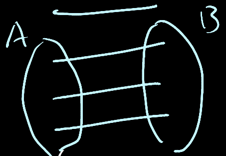
Figure: Thoralf Skolem

Figure: Abraham Fraenkel

Zermelo-Fraenkel Set Theory

替换

Replacement Schema: For each formula $\varphi(x, y, v_1, \ldots, v_k)$, the following is an axiom of replacement



$$\forall x \exists! y\, \varphi(x, y, v_1, \ldots, v_k)$$

$$\rightarrow \forall A \exists B \forall y (y \in B \leftrightarrow \exists x (x \in A \wedge \varphi(x, y, v_1, \ldots, v_k)))$$

B is the φ replacement of A

Zermelo-Fraenkel Set Theory

+ C

= ZFC

Axiom of choice (AC)

$$\forall A\Big(\forall X(X \in A \to X \neq \emptyset) \to \forall X \forall Y(X \in A \wedge Y \in A \wedge X \neq Y \to X \cap Y = \emptyset)$$

$$\to \exists B \forall X\big(X \in A \to \exists! z\; z \in B \cap X\big)\Big)$$

Zermelo-Fraenkel Set Theory

Axiom of choice

$$\forall A\Big(\forall X(X \in A \to X \neq \emptyset) \to \forall X \forall Y(X \in A \wedge Y \in A \wedge X \neq Y \to X \cap Y = \emptyset)$$
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Zermelo-Fraenkel Set Theory

Axiom of choice

$$\forall A\Big(\forall X(X \in A \to X \neq \emptyset) \to A \text{ is pairwise disjoint}$$
$$\to \exists B \forall X\big(X \in A \to \exists! z\; z \in B \cap X\big)\Big)$$

Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

Axiom of choice

$$\forall A \Big(\text{elements of } A \text{ are nonempty} \to A \text{ is pairwise disjoint} \\ \to \exists B \forall X \big(X \in A \to \exists! z \; z \in B \cap X \big) \Big)$$

Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

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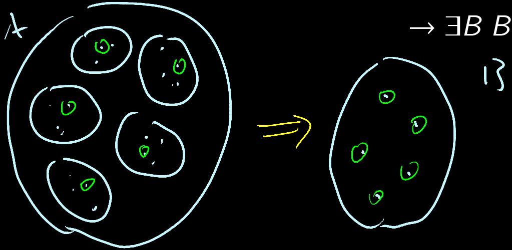
$$\forall A\Big(\text{elements of } A \text{ are nonempty} \rightarrow A \text{ is pairwise disjoint}$$
$$\rightarrow \exists B \forall X \Big(X \in A \rightarrow \exists z \; B \cap X = \{z\}\Big)\Big)$$

Zermelo-Fraenkel Set Theory

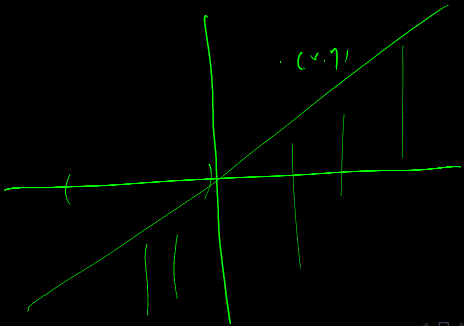
Axiom of choice

$$\forall A\Big(\text{elements of } A \text{ are nonempty} \to A \text{ is pairwise disjoint}$$

$$\to \exists B\ B \text{ is a choice set of } A\Big)$$



Relations and Functions



Relations and Functions

Definition

We define (x, y) to be $\{\{x\}, \{x, y\}\}$

Fact

- $\forall x \forall y \exists! Z \; Z = (x, y)$
- $(x_0, y_0) = (x_1, y_1)$ if and only if $x_0 = x_1$ and $y_0 = y_1$

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Relations and Functions

Definition

We define the Cartesian product of sets A, B to be

$$A \times B =_{\text{abbr}} \left\{ (a, b) \;\middle|\; a \in A \land b \in B \right\}$$

Fact

For any sets A and B, there exists a unique Cartesian product of A, B

Relations and Functions

$\{a\} \subseteq A$

$\{a, b\} \subseteq A \cup B \Rightarrow \{a, b\} \in \mathcal{P}(A \cup B)$

Definition

We define the Cartesian product of sets A, B to be

$\in \mathcal{P}\mathcal{P}(A \cup B)$

$$A \times B =_{\text{abbr}} \Big\{(a, b) \;\Big|\; a \in A \wedge b \in B\Big\}$$

$\{\{a\}, \{a, b\}\} \subseteq \mathcal{P}(A \cup B)$

$\in \mathcal{P}\mathcal{P}(A \cup B)$

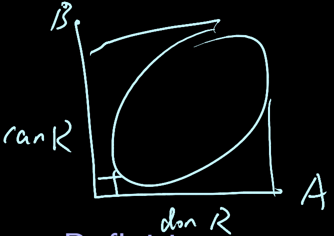
Fact

For any sets A and B, there exists a unique Cartesian product of A, B Exe

Relations and Functions

Figure: René Descartes (1596 - 1650)

Latinized: Renatus Cartesius



Relations and Functions

$R \subset A \times A =_{df} A^2$

Relation on A

Definition

- We say R is a (binary) relation, if $R \subset A \times B$ for some sets A, B. In this case, we also say R is a relation on $A \times B$.
- where the domain of R is $\operatorname{dom} R = \{a \mid \exists y\ R(a, y)\}$, the range of R is $\operatorname{ran} R = \{b \mid \exists x\ R(x, b)\}$

EXE: The domain of a relation is a set

Relations and Functions

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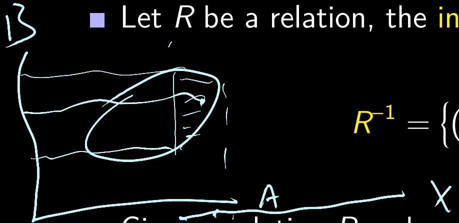
EXE: The domain of a relation is a set

Relations and Functions

Definition

- Let R be a relation, the inverse of R is defined to be

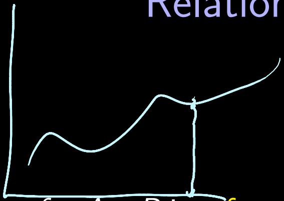


$$R^{-1} = \left\{ (b, a) \;\middle|\; R(a, b) \right\}$$

- Given a relation R and a set X, define the image of X under R to be the set

$$R[X] = R``X = \left\{ b \in \operatorname{ran} R \;\middle|\; \exists x \in X \; R(x, b) \right\}$$

Relations and Functions



Definition

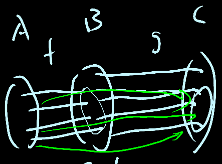
- A relation $f \subset A \times B$ is a function if for any $a \in \mathrm{dom}\, R$ there exists unique b such that $(a, b) \in f$. In this case, write $f(a)$ for the unique b such that $(a, b) \in f$.
- We say f is a function from A to B ($f : A \to B$) if $\mathrm{dom}\, f = A$ and $\mathrm{ran}\, f \subset B$

  $B^A \subseteq P(A \times B)$
- Define B^A to be the set of all function from A to B

  B^A exists $\forall f.\ f \in B^A \to f \subseteq A \times B$

  $f \in P(A \times B)$

Relations and Functions

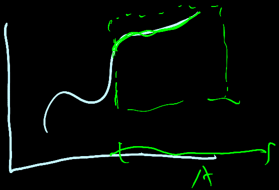


Definition

- Given $f: A \to B$ and $g: B \to C$, define

$$g \circ f = \left\{(a, c) \in A \times C \;\middle|\; \exists b \in B \ f(a) = b \wedge g(b) = c\right\}$$

- f be a function, A is a set. Define $f \restriction A = f \cap (A \times \operatorname{ran} f)$



Definition

- A function f is injective if for each $b \in \operatorname{ran} f$ there exists a unique $a \in \operatorname{dom} f$ such that $f(a) = b$
- We say $f : A \to B$ is surjective if $\operatorname{ran} f = B$
- $f : A \to B$ is bijective if f is both injective and surjective

Relations and Functions

Fact

- If f and g are functions, then $g \circ f$ is a function
- Let f be a injective function, then f^{-1} is a function and $(f^{-1})^{-1} = f$

Proposition (ZF): $AC \leftrightarrow$ for each every set A whose elements are nonempty, there is a choice function $f: A \to \bigcup A$ such that $f(a) \in a$ for all $a \in A$.

($\Leftarrow$) Let the choice set be $f[A]$

($\Rightarrow$) Fix A whose elements are nonempty

For each $a \in A$, define $D(a) = \{(b,a) \mid b \in a\}$

Let $A' = \{D(a) \mid a \in A\}$.

Check: A' is pairwise disjoint, each D(a) is nonempty ($a \in A$)

Let B be a choice set for A' (by AC)

Define $f = \{(a,b) \mid (b,a) \in B\}$

Check:

① f is a function [B is a choice set]

② dom f = A

For each $a \in A$, that is, for each $D(a) \in A'$, there is a $(b,a) \in B \cap D(a)$, so $(a,b) \in f$, thus $a \in \text{dom } f$

③ $f(a) = a$ for each $a \in A$

To see ③:

By definition of f, $(f(a), a) \in B$ for each $a \in A$

so $(f(a), a) \in \bigcup A'$, since $B \subseteq \bigcup A'$

so $(f(a), a) \in D(c)$ for some c,

but $a = c$, by the definition of $D(c)$, i.e.

$(f(a), a) \in D(a)$ i.e. $f(a) \in a$

by the definition of $D(a)$. □

Next on Set Theory

Orders and the theory of ordinals

Exercise

logic.fudan.edu.cn

1. Show that $R^{-1}[Y] = \left\{a \in \operatorname{dom} R \,\middle|\, \exists y \in Y\, R(a, y)\right\}$
2. Show that

   $\forall A$(elements of A are nonempty $\to$ A has a choice set) is false, so the requirement for A to be pairwise disjoint is necessary.
3. Show that $PP(\emptyset)$ exists (use only ZF − Power). $(*)$ Show that for each n, ZF − Power proves $P^n(\emptyset)$ exists.

Exercise

4. We say function f and g are compatible if $f(x) = g(x)$ for every $x \in \operatorname{dom} f \cap \operatorname{dom} g$; a set of function $\mathcal{F}$ is compatible if the functions in $\mathcal{F}$ are pairwise compatible.
   Show that if $\mathcal{F}$ is a compatible set of functions, then $\bigcup \mathcal{F}$ is a function.