Set Theory

集合论

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Previously on Set Theory

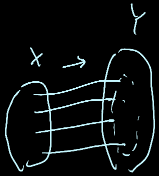
助教：

- 单芃舒（14210160029@fudan.edu.cn）
- 微信互助小组：集合论 2015 秋

习题、考试

- 自愿提交，习题课（正课课后）
- 开卷考试

Previously on Set Theory



- What set theory is about?
- How big is a set?
  - Cantor's definition of $\leq$ and $\sim$
  - Cantor Theorem: there is another infinity!
  - Cantor-Bernstein Theorem $X \leq Y, Y \leq X \to X \sim Y$

Set theory is about the sizes of sets (of reals).

Previously on Set Theory

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Set theory is about the sizes of sets (of reals).

Previously on Set Theory

- But what is a size/number? Caesar problem
- Frege's definition of number
- Frege's assumption and Russell's Paradox

Here comes the axiomatic set theory, ZFC

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Previously on Set Theory

$\exists z (z \in X)$

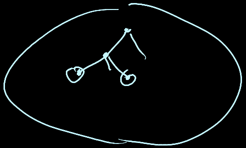
$$\forall X (X \neq \emptyset \rightarrow \exists y (y \in X \wedge y \cap X = \emptyset))$$

$\neg \exists z (z \in y \wedge z \in X)$

ZFC:

- Formal language
- Axiom of extensionality
- Axiom of foundation
- Axiom of pairs

$$\forall X \forall Y (X = Y \leftrightarrow \forall z (z \in X \leftrightarrow z \in Y))$$



$\{x, y\} = z$

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w = x \vee w = y)$$

Zermelo-Fraenkel Set Theory

Definition (Singleton)

We write $y \in \{x\}$ / $Y = \{x\}$ instead of $x \in \{y, y\}$ / $X = \{y, y\}$

Fact

$\forall x \exists Y \forall Y' (Y' = Y \leftrightarrow Y' = \{x\})$

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Zermelo-Fraenkel Set Theory

$y = x \vee y = x \quad \forall z (z \in Y \leftrightarrow z = x \vee z = x)$

$y \in \{x, x\} \ / \ Y = \{x, x\}$

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We write $y \in \{x\}$ / $Y = \{x\}$ instead of $x \in \{y, y\}$ / $X = \{y, y\}$

Fact

$Y = \{x\}$

$\forall X \exists! Y \ Y\{x\}$

Zermelo-Fraenkel Set Theory

$X = \{Y_0, Y_1, Y_2\}$

$z \in \bigcup X \leftrightarrow z \in Y_0 \vee z \in Y_1 \vee z \in Y_2$

Axiom of union

$$\forall X \exists Y y = \bigcup X$$

where $z \in \bigcup X$ is abbr. for $\exists Y (Y \in X \wedge z \in Y)$

Fact (Pair + Union)

$\forall X \forall Y \exists Z (Z = X \cup Y)$, where $X \cup Y = \bigcup \{X, Y\}$

Zermelo-Fraenkel Set Theory

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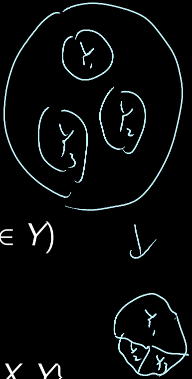

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Zermelo-Fraenkel Set Theory

Axiom of power set

$\forall w(w \in z \to w \in X)$

$$\forall X \exists Y \forall Z (Z \in Y \leftrightarrow Z \subset X)$$

Let $Z \in P(X)$ be an abbr. for $Z \subset X$, and $Y = P(X)$ for $\forall Z (Z \in Y \leftrightarrow Z \subset X)$

Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

What kind of set we have now?

We know we have some, but we don't know what we have yet

Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

$x \neq x$

Separation schema: For each formula of the language of set theory $\varphi(x, v_1, \ldots, v_k)$, the following is an axiom of separation

$$\forall A \exists B \forall x (x \in B \leftrightarrow x \in A \wedge \varphi(x, v_1, \ldots, v_k))$$

Let $B = \{x \in A \mid \varphi(x)\}$ be abbr. for $\forall x (x \in B \leftrightarrow x \in A \wedge \varphi(x))$

Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

Example

$\{x \in Y \mid x \neq x\} = \emptyset$

$R = \{x \in A \mid x \notin x\}$

$R \in R$?

- Empty set exists: $\exists x \; x = \emptyset$
- Intersection exists I: $\forall X \forall Y \exists Z \; Z = X \cap Y$

  where $Z = X \cap Y$ is abbr. for $\forall x (x \in Z \leftrightarrow z \in X \wedge x \in Y)$
- Intersection exists II: $\forall A (A \neq \emptyset \rightarrow \exists B \; B = \bigcap A)$

  where $B = \bigcap A$ is abbr. for

  $\forall x (x \in B \leftrightarrow \forall y (y \in A \rightarrow x \in y))$

Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

$x \in A$

Example

$\{z \in X \mid \forall Y (Y \in A \rightarrow z \in Y)\}$

- Empty set exists: $\exists x \; x = \emptyset$
- Intersection exists I: $\forall X \forall Y \exists Z \; Z = X \cap Y$ $\{z \in X \mid z \in Y\}$

  where $Z = X \cap Y$ is abbr. for $\forall x(x \in Z \leftrightarrow z \in X \wedge x \in Y)$
- Intersection exists II: $\forall A(A \neq \emptyset \rightarrow \exists B \; B = \bigcap A)$

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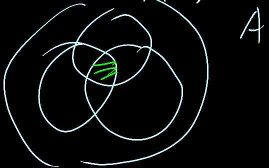

Zermelo-Fraenkel Set Theory

Example

- $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}, \{\{\{\emptyset\}\}\} \ldots$
- Define Sx (the successor of x) to be the set $x \cup \{x\}$

  EXE: Every set has a successor

  $\emptyset, S\emptyset = \{\emptyset\}, SS\emptyset = \{\emptyset, \{\emptyset\}\}, \ldots, S^n\emptyset, \ldots$
- $\emptyset, P(\emptyset), P(P(\emptyset)), \ldots, P^n(\emptyset) \ldots$

Zermelo-Fraenkel Set Theory

Example

$S\emptyset = \emptyset \cup \{\emptyset\} = \{\emptyset\}$

$S\{\emptyset\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$

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  $= \{\emptyset\}$ $= \{\emptyset, \{\emptyset\}\}$

Zermelo-Fraenkel Set Theory

Definition

When we say X is inductive, we mean

$$\emptyset \in X \wedge \forall y(y \in X \to Sy \in X)$$

Axiom of infinity

$$\exists X \ X \text{ is inductive}$$

Zermelo-Fraenkel Set Theory

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When we say X is inductive, we mean

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Axiom of infinity

$$\exists X \; X \text{ is inductive}$$

Zermelo-Fraenkel Set Theory

Definition

Define ω to be the "smallest inductive set", i.e. $x = \omega$ is abbr. for

$$x \text{ is inductive } \wedge \forall y (y \text{ is inductive } \rightarrow x \subset y)$$

Fact

ω is well-defined (unique) and exists.

Zermelo-Fraenkel Set Theory

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Define ω to be the "smallest inductive set", i.e. $x = \omega$ is abbr. for

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By Inf, there is an inductive set Y

Let $\omega = \{x \in Y \mid \forall X (X \text{ is inductive} \rightarrow x \in$

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ω is well-defined (unique) and exists.

Zermelo-Fraenkel Set Theory

Zermelo-Fraenkel Set Theory

Figure: Thoralf Skolem

Figure: Abraham Fraenkel

Zermelo-Fraenkel Set Theory

Replacement Schema: For each formula $\varphi(x, y, v_1, \ldots, v_k)$, the following is an axiom of replacement

$$\forall x \exists ! y \, \varphi(x, y, v_1, \ldots, v_k)$$
$$\rightarrow \forall A \exists B \forall y (y \in B \leftrightarrow \exists x (x \in A \wedge \varphi(x, y, v_1, \ldots, v_k)))$$

Zermelo-Fraenkel Set Theory

Axiom of choice

$$\forall A\Big(\forall X(X \in A \to X \neq \emptyset) \to \forall X \forall Y(X \in A \land Y \in A \land X \neq Y \to X \cap Y = \emptyset)$$
$$\to \exists B \forall X\Big(X \in A \to \exists! z\; z \in B \cap X\Big)\Big)$$

Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

Axiom of choice

$$\forall A\Big(\forall X(X \in A \to X \neq \emptyset) \to A \text{ is pairwise disjoint}$$
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Zermelo-Fraenkel Set Theory

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Zermelo-Fraenkel Set Theory

Axiom of choice

$$\forall A\Big(\text{elements of } A \text{ are nonempty} \rightarrow A \text{ is pairwise disjoint} \rightarrow \exists B \forall X\big(X \in A \rightarrow \exists! z\ z \in B \cap X\big)\Big)$$

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Zermelo-Fraenkel Set Theory

Axiom of choice

$$\forall A\Big(\text{elements of } A \text{ are nonempty} \rightarrow A \text{ is pairwise disjoint}$$
$$\rightarrow \exists B\ B \text{ is a choice set of } A\Big)$$

Relations and Functions

Relations and Functions

Definition

We define (x, y) to be $\{\{x\}, \{x, y\}\}$

Fact

- $\forall x \forall y \exists! Z \; Z = (x, y)$
- $(x_0, y_0) = (x_1, y_1)$ if and only if $x_0 = x_1$ and $y_0 = y_1$

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Relations and Functions

Definition

We define the Cartesian product of sets A, B to be

$$A \times B =_{\text{abbr}} \left\{(a, b) \;\middle|\; a \in A \wedge b \in B\right\}$$

Fact

For any sets A and B, there exists a unique Cartesian product of A, B

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Relations and Functions

Figure: René Descartes (1596 - 1650)

Latinized: Renatus Cartesius

Relations and Functions

Definition

- We say R is a (binary) relation, if $R \subset A \times B$ for some sets A, B. In this case, we also say R is a relation on $A \times B$.
- where the domain of R is $\text{dom}\, R = \{a \mid \exists y\, R(a, y)\}$, the range of R is $\text{ran}\, R = \{b \mid \exists x\, R(x, b)\}$

EXE: The domain of a relation is a set

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EXE: The domain of a relation is a set

Relations and Functions

Definition

- Let R be a relation, the inverse of R is defined to be

$$R^{-1} = \left\{(b,a) \;\middle|\; R(a,b)\right\}$$

- Given a relation R and a set X, define the image of X under R to be the set

$$R[X] = R``X = \left\{b \in \operatorname{ran} R \;\middle|\; \exists x \in X\ R(x,b)\right\}$$

Relations and Functions

Definition

- A relation $f \subset A \times B$ is a function if for any $a \in \operatorname{dom} R$ there exists unique b such that $(a, b) \in f$. In this case, write $f(a)$ for the unique b such that $(a, b) \in f$.
- We say f is a function from A to B ($f : A \to B$) if $\operatorname{dom} f = A$ and $\operatorname{ran} f \subset B$
- Define B^A to be the set of all function from A to B

Relations and Functions

Definition

- Given $f : A \to B$ and $g : B \to C$, define

  $g \circ f = \left\{(a, c) \in A \times C \,\middle|\, \exists b \in B\ f(a) = b \wedge g(b) = c\right\}$

- f be a function, A is a set. Define $f \restriction A = f \cap (A \times \operatorname{ran} f)$

Relations and Functions

Definition

- A function f is injective if for each $b \in \operatorname{ran} f$ there exists a unique $a \in \operatorname{dom} f$ such that $f(a) = b$
- We say $f : A \to B$ is surjective if $\operatorname{ran} f = B$
- $f : A \to B$ is bijective if f is both injective and surjective

Relations and Functions

Fact

- If f and g are functions, then $g \circ f$ is a function
- Let f be a injective function, then f^{-1} is a function and $(f^{-1})^{-1} = f$

Proposition (ZF): $AC \leftrightarrow$ for each every set A whose elements are nonempty, there is a **choice function** $f : A \to \bigcup A$ such that $f(a) \in a$ for all $x \in A$.

Next on Set Theory

Orders and the theory of ordinals

Exercise

1. Show that $R^{-1}[Y] = \left\{ a \in \operatorname{dom} R \,\middle|\, \exists y \in Y\, R(a,y) \right\}$
2. Show that

   $\forall A(\text{elements of } A \text{ are nonempty} \to A \text{ has a choice set})$ is false, so the requirement for A to be pairwise disjoint is necessary.
3. Show that $PP(\emptyset)$ exists (use only $\mathrm{ZF} - \mathrm{Power}$). $(*)$ Show that for each n, $\mathrm{ZF} - \mathrm{Power}$ proves $P^n(\emptyset)$ exists.

Exercise

4. We say function f and g are compatible if $f(x) = g(x)$ for every $x \in \operatorname{dom} f \cap \operatorname{dom} g$; a set of function $\mathcal{F}$ is compatible if the functions in $\mathcal{F}$ are pairwise compatible.

   Show that if $\mathcal{F}$ is a compatible set of functions, then $\bigcup \mathcal{F}$ is a function.