

Set Theory

# 集合论

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# 助教团队

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Figure: 微信互助小组

# 作业、习题课、考试

What set theory is about?

# What set theory is about?

- If  $X \subset Y$  and  $Y \subset Z$ , then  $X \subset Z$
- $X \cap Y = Y \cap X$
- $(X \cup Y) \cup Z = X \cup (Y \cup Z)$
- $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$



You may think these are all what set theory is about.

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You may think these are all what set theory is about.

# What set theory is about?

- $(p \rightarrow q \wedge q \rightarrow r) \rightarrow p \rightarrow r$
- $p \wedge q \leftrightarrow q \wedge p$
- $(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$
- $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$

But these are more about logic rather than sets.

EXE: prove the laws of set theory using the laws of logic.



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But these are more about logic rather than sets.

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# The birth of set theory

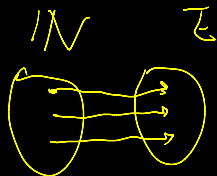


Figure: Georg Cantor (1845-1918)

# The birth of set theory

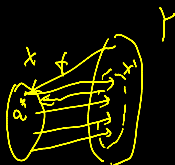
How big is a set?

$$n \rightarrow 2^n$$
$$\mathbb{N} \rightarrow \mathbb{E}$$



# The birth of set theory

$$a \in X$$
$$f(x) = a \quad \begin{matrix} x \in \text{ran } f \\ x \notin \text{ran } f \end{matrix}$$



One-to-one correspondence

Let  $X$  and  $Y$  be arbitrary sets.

Define "the size of  $X$  is no bigger than  $Y$ " (written  $X \leq Y$ ) if there is an **injection**  $f: X \rightarrow Y$ , i.e. there is a surjection  $g: Y \rightarrow X$ .

Define " $X$  is of the same size as  $Y$ " ( $X \sim Y$ ) if there is a **bijection**  $f: X \rightarrow Y$ .

# The birth of set theory

$f: X \rightarrow Y$  is an injection if  $f(x) = f(y) \Rightarrow x = y$   
surjection if  $x \neq y \Rightarrow f(x) \neq f(y)$



One-to-one correspondence

Let  $X$  and  $Y$  be arbitrary sets.

if  $\forall x \in X \exists y \in Y$  is for every  $y \in Y$  there is  $x \in X$  s.t.

Define "the size of  $X$  is no bigger than  $Y$ " (written  $X \leq Y$ ) if

there is an **injection**  $f: X \rightarrow Y$

$$f(x) = y$$

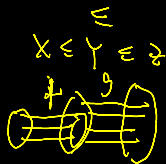
Define "X is of the same size of Y" ( $X \sim Y$ ) if there is a

bijection  $f: X \rightarrow Y$

injection and surjection



# The birth of set theory



$$g(f(x)) = h(x) = g \circ f(x)$$

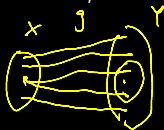
Fact

- $\leq$  and  $\sim$  are transitive
- If  $X \leq Y$ , then there is a surjection  $g: Y \rightarrow X$
- If there is a surjection  $g: Y \rightarrow X$ , then  $X \leq Y$  (AC)

$v \in X$

$$Y_x = \{y \mid g(y) = x\}$$

$f(x) = \text{pick one in } Y_x$



# The birth of set theory

## Fact

- the set of algebraic numbers  $\leq \mathbb{Q} \leq \mathbb{Z}^2 \sim \mathbb{Z} \sim \mathbb{N} \sim \mathbb{N}^2$
- $\mathbb{R} \sim (0, 1)$

# The birth of set theory

Theorem (Cantor)

$\mathbb{N} \leq \mathbb{R}$ , but  $\mathbb{R} \not\leq \mathbb{N}$  (written  $\mathbb{N} < \mathbb{R}$ )



# Theorem

$\mathbb{N} \subset \mathbb{R}$

1)  $\mathbb{N} \subseteq \mathbb{R}$   $f(n) = n$  as real for  $n \in \mathbb{N}$

2)  $\mathbb{R} \subseteq \mathbb{N}$ , Assume there is  $g: \mathbb{N} \rightarrow \mathbb{R}$  s.t.  $\text{rang } g = \mathbb{R}$

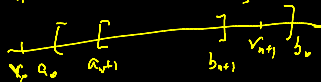
then  $\mathbb{R} = \{r_0, r_1, r_2, \dots\}$  where  $r_n = g(n)$  for  $n \in \mathbb{N}$

Find  $a_0, b_0$  s.t.  $a_0 < b_0$  and  $r_0 \in [a_0, b_0]$

For <sup>step</sup>  $n+1$  Given  $a_0, \dots, a_n, b_0, \dots, b_n$

Find  $a_{n+1}, b_{n+1}$  s.t.  $a_n < a_{n+1} < b_{n+1} < b_n$  s.t.  $r_{n+1} \in [a_{n+1}, b_{n+1}]$

and  $b_n - a_{n+1} < \frac{1}{n+1}$



Let  $r = \sup\{a_0, a_1, \dots\}$

claim  $r \neq r_n$  for  $n \in \mathbb{N}$

subclaim  $r \in [a_n, b_n]$  for  $n \in \mathbb{N}$

$r < b_n$ , for otherwise  $b_n < r$   $\rightarrow \infty$

## Theorem

$$\mathbb{N} < \mathbb{R}$$

# The birth of set theory

Theorem (Cantor-Bernstein)

If  $X \leq Y$  and  $Y \leq X$ , then  $X \sim Y$

# Theorem (Cantor-Bernstein)

If  $X \leq Y$  and  $Y \leq X$ , then  $X \sim Y$

have:  $f: X \rightarrow Y$ ,  $g: Y \rightarrow X$ , want to build  $h: X \rightarrow Y$

Define for  $x \in X$ , define  $x^{(0)} = x$



## Theorem(Cantor-Bernstein)

If  $X \leq Y$  and  $Y \leq X$ , then  $X \sim Y$

# Continuum Hypothesis (CH)

There is no set  $A$  such that

$$\aleph_1 < A < \aleph_2$$

Cantor's Continuum Problem (1st Hilbert Problem):

Is CH true or false?

Some even say: set theory is about sets of reals.

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Some even say: set theory is about **the sizes of sets of reals**.

# What is a number?

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# What a number is

Frege:

The number 0 is the set of all sets which has the same size as  $\{x \mid x \neq x\}$ , i.e.

$$0 = \{X \mid X \sim \{x \mid x \neq x\}\}$$

The number of  $\mathbb{R}$  is

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Frege's assumption:

Each concept  $\varphi(x)$  has an extension, that is the set

$$\{x \mid \varphi(x)\}$$

Russell: Consider the set

$$R = \{x \mid x \notin x\}$$



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# Russell's Paradox

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Russell: Consider the set

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# Zermelo-Fraenkel Set Theory



Figure: Ernst Zermelo (1871 - 1953)

# Zermelo-Fraenkel Set Theory

The language of set theory:  $\mathcal{L}_\in$

- **variables:**  $x, y, z, \dots, X, Y, Z, \dots a, b, c, \dots x_1, x_2, \dots$
- **connectives:**  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- **quantifiers:**  $\forall x, \forall y, \dots, \exists x, \exists y, \dots$
- **relation sign:**  $=, \in$
- **parentheses:**  $(, )$

# Zermelo-Fraenkel Set Theory

Axiom of existence

$$\exists x(x = x)$$

Axiom of extensionality

$$\forall X \forall Y \forall z (z \in X \rightarrow z \in Y) \rightarrow X = Y$$

# Zermelo-Fraenkel Set Theory

## Definition

When we say “ $X$  is a **subset** of  $Y$ ” (abbr.  $X \subset Y$ ), we mean

$$\forall z(z \in X \rightarrow z \in Y)$$

“ $X$  is a proper subset of  $Y$ ” (abbr.  $X \subsetneq Y$ ) is defined to be

$$X \subset Y \wedge X \neq Y$$

# Zermelo-Fraenkel Set Theory

## Definition

We write  $x = \emptyset$  for  $\neg \exists y y \in x$

# Zermelo-Fraenkel Set Theory

Axiom of foundation

$$\forall X(X \neq \emptyset \rightarrow \exists y(y \in X \wedge \forall z(z \in X \rightarrow z \notin y)))$$

Definition

$z \in X \cap Y$  is abbr. for  $z \in X \wedge z \in Y$ .

EXE:  $\forall z(z \in X \rightarrow z \notin y) \leftrightarrow X \cap y = \emptyset$



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# Zermelo-Fraenkel Set Theory

Axiom of pair

$$\forall x \forall y \exists Z Z = \{x, y\}$$

where  $z \in \{x, y\}$  stands for  $z = x \vee z = y$ .

Fact (Pair + Foundation)

$$x \notin x$$

# Zermelo-Fraenkel Set Theory

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# Zermelo-Fraenkel Set Theory

Axiom of union

$$\forall X \exists Y \forall z (z \in Y \iff \exists X (z \in X))$$

where  $z \in \bigcup X$  is abbr. for  $\exists Y (Y \in X \wedge z \in Y)$

Fact (Pair + Union)

$\forall X \forall Y \exists Z (Z = X \cup Y)$ , where  $X \cup Y = \bigcup \{X, Y\}$

# Zermelo-Fraenkel Set Theory

Axiom of union

$$\forall X \exists Y \forall z (z \in Y \iff \exists X (X \in X \wedge z \in X))$$

where  $z \in \bigcup X$  is abbr. for  $\exists Y (Y \in X \wedge z \in Y)$

Fact (Pair + Union)

$\forall X \forall Y \exists Z (Z = X \cup Y)$ , where  $X \cup Y = \bigcup \{X, Y\}$

## Exercise

1. Show that if  $X$  is a proper subset of  $Y$ , then there is some  $z \in Y$  but  $z \notin X$ .
- 2.\* Show that  $\mathbb{R} \sim \{0, 1\}^{\mathbb{N}} \sim \mathbb{N}^{\mathbb{N}}$  if you know what it means.
3. Write down the axiom of pairs with no abbreviation!
4. Show that for each  $k \in \mathbb{N}$  and sets  $a_1, \dots, a_k$ , there is a unique set  $\{a_1, \dots, a_k\}$ . What do you presume in your prove?