

Set Theory

集合论

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Figure: 微信互助小组

作业、习题课、考试

What set theory is about?

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- If $X \subset Y$ and $Y \subset Z$, then $X \subset Z$
- $X \cap Y = Y \cap X$
- $(X \cup Y) \cup Z = X \cup (Y \cup Z)$
- $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$



You may think these are all what set theory is about.

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- $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

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What set theory is about?

- $(p \rightarrow q \wedge q \rightarrow r) \rightarrow p \rightarrow r$
- $p \wedge q \leftrightarrow q \wedge p$
- $(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$
- $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$

But these are more about logic rather than sets.

EXE: prove the laws of set theory using the laws of logic.

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The birth of set theory

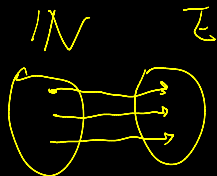


Figure: Georg Cantor (1845-1918)

The birth of set theory

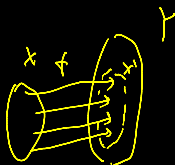
How big is a set?

$$n \rightarrow 2^n$$
$$\mathbb{N} \rightarrow \mathbb{E}$$



a

The birth of set theory



One-to-one correspondence

Let X and Y be arbitrary sets.

Define “the size of X is no bigger than Y ” (written $X \leq Y$) if there is an **injection** $f: X \rightarrow Y$, i.e. there is a surjection $g: Y \rightarrow X$.

Define “ X is of the same size of Y ” ($X \sim Y$) if there is a **bijection** $f: X \rightarrow Y$.

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The birth of set theory

Fact

- \leq and \sim are transitive
- If $X \leq Y$, then there is a surjection $g: Y \rightarrow X$
- If there is a surjection $g: Y \rightarrow X$, then $X \leq Y$ (AC)

The birth of set theory

Fact

- the set of algebraic numbers $\leq \mathbb{Q} \leq \mathbb{Z}^2 \sim \mathbb{Z} \sim \mathbb{N} \sim \mathbb{N}^2$
- $\mathbb{R} \sim (0, 1)$

The birth of set theory

Theorem (Cantor)

$\mathbb{N} \leq \mathbb{R}$, but $\mathbb{R} \not\leq \mathbb{N}$ (written $\mathbb{N} < \mathbb{R}$)

Theorem

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The birth of set theory

Theorem (Cantor-Bernstein)

If $X \leq Y$ and $Y \leq X$, then $X \sim Y$

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Continuum Hypothesis (CH)

There is no set A such that

$$\aleph_1 < A < \aleph_2$$

Cantor's Continuum Problem (1st Hilbert Problem):

Is CH true or false?

Some even say: set theory is about sets of reals.

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Some even say: set theory is about **the sizes of sets of reals**.

What is a number?

Caesar problem: Is the number of “the students in the classroom”/ℝ Julius Caesar?

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What a number is

Frege:

The number 0 is the set of all sets which has the same size as $\{x \mid x \neq x\}$, i.e.

$$0 = \{X \mid X \sim \{x \mid x \neq x\}\}$$

The number of \mathbb{R} is

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Frege's assumption:

Each concept $\varphi(x)$ has an extension, that is the set

$$\{x \mid \varphi(x)\}$$

Russell: Consider the set

$$R = \{x \mid x \notin x\}$$

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Russell's Paradox

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Zermelo-Fraenkel Set Theory



Figure: Ernst Zermelo (1871 - 1953)

Zermelo-Fraenkel Set Theory

The language of set theory: \mathcal{L}_\in

- **variables:** $x, y, z, \dots, X, Y, Z, \dots a, b, c, \dots x_1, x_2, \dots$
- **connectives:** $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- **quantifiers:** $\forall x, \forall y, \dots, \exists x, \exists y, \dots$
- **relation sign:** $=, \in$
- **parentheses:** $(,)$

Zermelo-Fraenkel Set Theory

Axiom of existence

$$\exists x(x = x)$$

Axiom of extensionality

$$\forall X \forall Y \forall z (z \in X \rightarrow z \in Y) \rightarrow X = Y$$

Zermelo-Fraenkel Set Theory

Definition

When we say “ X is a **subset** of Y ” (abbr. $X \subset Y$), we mean

$$\forall z(z \in X \rightarrow z \in Y)$$

“ X is a proper subset of Y ” (abbr. $X \subsetneq Y$) is defined to be

$$X \subset Y \wedge X \neq Y$$

Zermelo-Fraenkel Set Theory

Definition

We write $x = \emptyset$ for $\neg \exists y y \in x$

Zermelo-Fraenkel Set Theory

Axiom of foundation

$$\forall X(X \neq \emptyset \rightarrow \exists y(y \in X \wedge \forall z(z \in X \rightarrow z \notin y)))$$

Definition

$z \in X \cap Y$ is abbr. for $z \in X \wedge z \in Y$.

EXE: $\forall z(z \in X \rightarrow z \notin y) \leftrightarrow X \cap y = \emptyset$

Zermelo-Fraenkel Set Theory

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EXE: $\forall z (z \in X \rightarrow z \notin y) \leftrightarrow X \cap y = \emptyset$

Zermelo-Fraenkel Set Theory

Axiom of pair

$$\forall x \forall y \exists Z Z = \{x, y\}$$

where $z \in \{x, y\}$ stands for $z = x \vee z = y$.

Fact (Pair + Foundation)

$$x \notin x$$

Zermelo-Fraenkel Set Theory

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$$\forall x \forall y \exists Z Z = \{x, y\}$$

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Fact (Pair + Foundation)

$$x \notin x$$

Zermelo-Fraenkel Set Theory

Axiom of union

$$\forall X \exists Y \forall z (z \in Y \iff \exists X (X \in X \wedge z \in X))$$

where $z \in \bigcup X$ is abbr. for $\exists Y (Y \in X \wedge z \in Y)$

Fact (Pair + Union)

$\forall X \forall Y \exists Z (Z = X \cup Y)$, where $X \cup Y = \bigcup \{X, Y\}$

Zermelo-Fraenkel Set Theory

Axiom of union

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$\forall X \forall Y \exists Z (Z = X \cup Y)$, where $X \cup Y = \bigcup \{X, Y\}$

Exercise

1. Show that if X is a proper subset of Y , then there is some $z \in Y$ but $z \notin X$.
- 2.* Show that $\mathbb{R} \sim \{0, 1\}^{\mathbb{N}} \sim \mathbb{N}^{\mathbb{N}}$ if you know what it means.
3. Write down the axiom of pairs with no abbreviation!
4. Show that for each $k \in \mathbb{N}$ and sets a_1, \dots, a_k , there is a unique set $\{a_1, \dots, a_k\}$. What do you presume in your prove?