Set Theory

集合论

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Figure: 微信互助小组

作业、习题课、考试

What set theory is about?

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- If $X \subset Y$ and $Y \subset Z$, then $X \subset Z$
- $X \cap Y = Y \cap X$
- $(X \cup Y) \cup Z = X \cup (Y \cup Z)$
- $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

You may think these are all what set theory is about.

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You may think these are all what set theory is about.

What set theory is about?

- $(p \to q \land q \to r) \to p \to r$
- $p \land q \leftrightarrow q \land p$
- $(p \lor q) \lor r \leftrightarrow p \lor (q \lor r)$
- $p \land (q \lor r) \leftrightarrow (p \land q) \lor (p \land r)$

But these are more about logic rather than sets.

EXE: prove the laws of set theory using the laws of logic.

What set theory is about?

- $(p \to q \wedge q \to r) \to p \to r$
- $p \wedge q \leftrightarrow q \wedge p$
- $(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$
- $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$

But these are more about logic rather than sets.

EXE: prove the laws of set theory using the laws of logic.

The birth of set theory

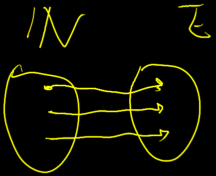
Figure: Georg Cantor (1845-1918)

The birth of set theory

How big is a set?

$n \to 2n$

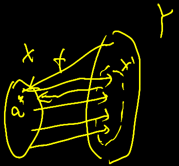
$\mathbb{N} \to E$



$a \in X$

$$g(x) = \begin{cases} f^{-1}(x) & x \in \operatorname{ran} f \\ a & x \notin \operatorname{ran} f \end{cases}$$

The birth of set theory



One-to-one correspondence

Let X an Y be arbitrary sets.

Define "the size of X is no bigger than Y" (written $X \leq Y$) if there is an injection $f : X \to Y$, i.e. there is a surjection $g : Y \to X$

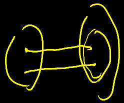
Define "X is of the same size of Y" ($X \sim Y$) if there is a bijection $f : X \to Y$

The birth of set theory

$f: X \to Y$ is an injection if $f(x) = f(y) \Rightarrow x = y$

surjection

$x \neq y \Rightarrow f(x) \neq f(y)$



One-to-one correspondence

if $\operatorname{ran} f = Y$ i.e. for all $y \in Y$ there is $x \in X$ s.t. $f(x) = y$

Let X an Y be arbitrary sets.

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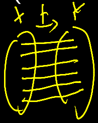
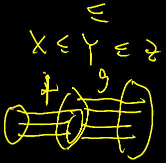
injection and surjection



The birth of set theory



$g(f(x)) = h(x) = g \circ f(x)$

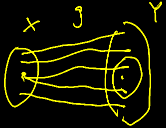
Fact

- $\leq$ and $\sim$ are transitive
- If $X \leq Y$, then there is a surjection $g : Y \to X$
- If there is a surjection $g : Y \to X$, then $X \leq Y$ (AC)

$x \in X$

$Y_x = \{y \mid g(y) = x\}$

$f(x) =$ pick one in Y_x



The birth of set theory

Fact

- the set of algebraic numbers $\leq \mathbb{Q} \leq \mathbb{Z}^2 \sim \mathbb{Z} \sim \mathbb{N} \sim \mathbb{N}^2$
- $\mathbb{R} \sim (0, 1)$

The birth of set theory

Theorem (Cantor)

$\mathbb{N} \leq \mathbb{R}$, but $\mathbb{R} \not\leq \mathbb{N}$ (written $\mathbb{N} < \mathbb{R}$)

Theorem

$\mathbb{N} < \mathbb{R}$

1) $\mathbb{N} \le \mathbb{R}$ $\quad f(n) = n$ as real for $n \in \mathbb{N}$

2). $\mathbb{R} \not\le \mathbb{N}$, Assume there is $g : \mathbb{N} \to \mathbb{R}$ s.t. $\operatorname{ran} g = \mathbb{R}$

then $\mathbb{R} = \{ r_0, r_1, r_2 \cdots \}$ where $r_n = g(n)$ for $n \in \mathbb{N}$

Find a_0, b_0 s.t. $a_0 < b_0$ and $r_0 \notin [a_0, b_0]$

For step $n+1$ Given $a_0 \cdots a_n$, $b_0 \cdots b_n$

Find a_{n+1}, b_{n+1} s.t. $a_n < a_{n+1} < b_{n+1} < b_n$ s.t. $r_{n+1} \notin [a_{n+1}, b_{n+1}]$

and $b_{n+1} - a_{n+1} < \frac{1}{n+1}$

Let $r = \sup \langle a_0, a_1, \cdots \rangle$

claim $r \ne r_n$ for $n \in \mathbb{N}$

subclaim $r \in [a_n, b_n]$ for $n \in \mathbb{N}$

$r < b_n$, for otherwise $b_n < r$ [illegible]

Theorem

$\mathbb{N} < \mathbb{R}$

The birth of set theory

Theorem (Cantor-Bernstein)

If $X \leq Y$ and $Y \leq X$, then $X \sim Y$

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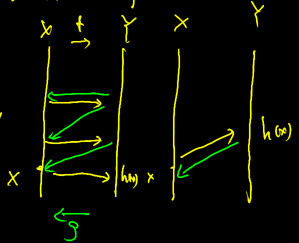
If $X \leq Y$ and $Y \leq X$, then $X \sim Y$

have: $f: X \xrightarrow{1-1} Y$, $g: Y \xrightarrow{1-1} X$, want to build $h: X \xleftrightarrow{1-1} Y$



<u>Define</u> for $x \in X$, let

$o(x)$ = the number $\{g^{-1}(x), f^{-1} \circ g^{-1}(x), g^{-1} f^{-1} g^{-1}(x) \ldots\}$

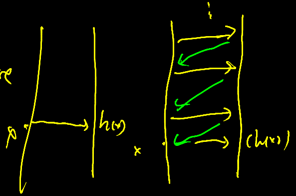
$X_e = \{x \mid o(x)$ is finite and even $\}$

$X_o = \{x \mid o(x)$ is finite and odd $\}$

$X_\infty = \{x \mid o(x)$ is $\infty\}$

then $X = X_e \cup X_o \cup X_\infty$ and X_e, X_o, X_∞ are pairwise disjoint

define Y_e, Y_o, Y_∞, similarly



Theorem (Cantor-Bernstein)

If $X \leq Y$ and $Y \leq X$, then $X \sim Y$

Def $h(x) = \begin{cases} f(x) & \text{if } x \in X_e \cup X_\infty \\ g^{-1}(x) & \text{if } x \in X_o \end{cases}$

Claim

h is injection

Since $h \restriction X_\infty$, $h \restriction X_o$, $h \restriction X_e$ are 1-1.

and $h''X_\infty = Y_\infty$, $h''X_e = Y_o$, $h''X_o = Y_e$ are pairwise disjoint

Claim h is surjection

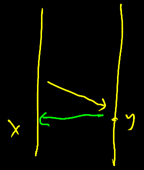
Assume $y \in$ ran h

Let $x = g(y)$

If $x \in X_\infty$, then $y \in Y_\infty$, then there is x', s.t. $f(x') = y$, and $x' \in X_\infty$, so $h(x') = y$

If $x \in X_o$, then $h(x) = g^{-1}(x) = y$

If $x \in X_e$, then $y \in Y_o$, there is $x' \in X_e$, and $f(x') = y$ then $h(x') = y$ □



Continuum Hypothesis (CH)

There is no set A such that

$$\mathbb{N} < A < \mathbb{R}$$

Cantor's Continuum Problem (1st Hilbert Problem):

Is CH true or false?

Some even say: set theory is about sets of reals.

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Some even say: set theory is about the sizes of sets of reals.

What is a number?

Caesar problem: Is the number of "the students in the classroom" $/\mathbb{R}$ Julius Caesar?

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What a number is

Frege:

The number 0 is the set of all sets which has the same size as $\{x \mid x \neq x\}$, i.e.

$$0 = \left\{X \;\middle|\; X \sim \{x \mid x \neq x\}\right\}$$

The number of $\mathbb{R}$ is

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Frege's assumption:

Each concept $\varphi(x)$ has an extension, that is the set

$$\{x \mid \varphi(x)\}$$

Russell: Consider the set

$$R = \{x \mid x \notin x\}$$

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$R \in R$?

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Russell's Paradox

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Zermelo-Fraenkel Set Theory

Figure: Ernst Zermelo (1871 - 1953)

Zermelo-Fraenkel Set Theory

The language of set theory: $\mathcal{L}_\in$

- variables: $x, y, z, \ldots, X, Y, Z, \ldots a, b, c, \ldots x_1, x_2, \ldots$
- connectives: $\neg, \wedge, \vee \;\rightarrow, \leftrightarrow$
- quantifiers: $\forall x, \forall y, \ldots, \exists x, \exists y, \ldots$
- relation sign: $=, \in$
- parentheses: $(,)$

Zermelo-Fraenkel Set Theory

Axiom of existence

$$\exists x(x = x)$$

Axiom of extensionality

$$\forall X \forall Y \forall z(z \in X \leftrightarrow z \in Y) \rightarrow X = Y$$

Zermelo-Fraenkel Set Theory

Definition

When we say "X is a subset of Y" (abbr. $X \subset Y$), we mean

$$\forall z(z \in X \rightarrow z \in Y)$$

"X is a proper subset of Y" (abbr. $X \subsetneq Y$) is defined to be

$$X \subset Y \wedge X \neq Y$$

Zermelo-Fraenkel Set Theory

Definition

$\neg \exists y \; y \in x$

We write $x = \emptyset$ for ~~$\neg y \; y \in x$~~

Zermelo-Fraenkel Set Theory

Axiom of foundation

$$\forall X\big(X \neq \emptyset \to \exists y(y \in X \wedge \forall z(z \in X \to z \notin y)\big)$$

Definition

$z \in X \cap Y$ is abbr. for $z \in X \wedge z \in Y$.

EXE: $\forall z(z \in X \to z \notin y) \leftrightarrow X \cap y = \emptyset$

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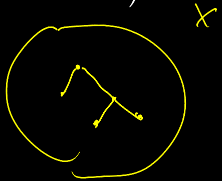

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EXE: $\forall z(z \in X \to z \notin y) \leftrightarrow X \cap y = \emptyset$

Zermelo-Fraenkel Set Theory

Axiom of pair

$$\forall x \forall y \exists Z \; Z = \{x, y\}$$

where $z \in \{x, y\}$ stands for $z = x \vee z = y$.

Fact (Pair + Fundation)

$x \notin x$

Zermelo-Fraenkel Set Theory

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where $z \in \{x, y\}$ stands for $z = x \vee z = y$.

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Zermelo-Fraenkel Set Theory

Axiom of union

$$\forall X \exists Y y = \bigcup X$$

where $z \in \bigcup X$ is abbr. for $\exists Y(Y \in X \wedge z \in Y)$

Fact (Pair + Union)

$\forall X \forall Y \exists Z(Z = X \cup Y)$, where $X \cup Y = \bigcup\{X, Y\}$

Zermelo-Fraenkel Set Theory

Axiom of union

$$\forall X \exists Y y = \bigcup X$$

where $z \in \bigcup X$ is abbr. for $\exists Y (Y \in X \land z \in Y)$

Fact (Pair + Union)

$\forall X \forall Y \exists Z (Z = X \cup Y)$, where $X \cup Y = \bigcup \{X, Y\}$

Exercise

1. Show that if X is a proper subset of Y, then there is some $z \in Y$ but $z \notin X$.
2.* Show that $\mathbb{R} \sim \{0,1\}^{\mathbb{N}} \sim \mathbb{N}^{\mathbb{N}}$ if you know what it means.
3. Write down the axiom of pairs with no abbreviationl.
4. Show that for each $k \in \mathbb{N}$ and sets $a_1, \ldots, a_k$, there is a unique set $\{a_1, \ldots, a_k\}$. What do you presume in your prove?