Infinite Injury

~~There is a uniform method to divide a set A into~~

We can divide ω into infinitely many sections: $\omega = \bigcup_e \omega^{[e]}$

For each set A, define $A^{[e]} = A \cap \omega^{[e]}$

Def We say a set B is piecewise recursive if $B^{[y]}$ is rec for each $y \in \omega$

Def We say $A \subseteq B$ is a thick subset if $B^{[y]} \subseteq^* A^{[y]}$ for all $y \in \omega$.

Thickness Lemma ~~(Shoe~~e

Let C be nonrec, r.e. ~~piecewise rec~~. B be a piecewise-rec r.e. set.

Then there is a thick r.e. subset A of B s.t. $C \not\le_T A$.

Proof Let $\{B_s\}_{s\in\omega}$, $\{C_s\}_{s\in\omega}$ be an enumeration of B and C respectively.

~~To generate A~~ We should generate A to meet the following requirement:

Positive: $P_e: B^{[e]} \subseteq^* A^{[e]}$

Negative: $N_e: C \ne \Phi_e^A$ (To preserve agreement $(C_s(y) = \Phi_{e,s}^{A_s}(y){\downarrow})$ as far as possible, so that if $C = \Phi_e^A$, then C is computable)

Note that for each e, P_e can not be met once for all, rather that it ~~should~~ may have to be treated infinitely often. And if P_i can injure N_e $(i<e)$, then N_e may be injured infinitely many times.

To generate A:

Let $A_0 = \phi$.

For the stage $s+1$. ~~For each $x \in B^{[e]}_{s+1}$, if~~ Given A_s defined.

Define:

$$a_s = \begin{cases} \mu x\,(x \in A_s - A_{s-1}) & \text{if } A_s - A_{s-1} \neq \phi \quad (\text{[illegible]急下降}) \\ \max(A_s \cup \{s\}) & \text{otherwise} \quad (\text{往上不切}) \end{cases}$$

Note a_s is computable from s.

$$\hat{\Phi}^{A_s}_{e,s}(x) = \begin{cases} \Phi^{A_s}_{e,s}(x) & \text{if } \Phi^{A_s}_{e,s}(x)\downarrow \text{ and } use\,\Phi^{A_s}_{e,s}(x) \leq a_s \\ \uparrow & \text{otherwise} \end{cases}$$

Note that, if $\Phi^A_e(x)\downarrow = y$

Then

$$\Phi^A_e(x) = \lim_s \Phi^{A_s}_{e,s}(x) = \lim_s \hat{\Phi}^{A_s}_{e,s}(x)$$

$\because$ Find s s.t. $\Phi^{A_s}_{e,s}(x)\downarrow = \Phi^A_e(x)$

and $A_s \upharpoonright use\,\Phi^{A}_{e}(x) = A \upharpoonright use\,\Phi^A_e(x)$

Then for all $t \geq s$, $a_t > use\,\Phi^{A_s}_{e,s}(x) = use\,\Phi^{A_t}_{e,t}(x)$

$\therefore \hat{\Phi}^{A_t}_{e,t}(x) = \Phi^{A_t}_{e,t}(x)$

Fact:

$$use\,\hat{\Phi}^{A_s}_{e,s}(x) = \begin{cases} use\,\Phi^{A_s}_{e,s}(x) & \text{if } \hat{\Phi}^{A_s}_{e,s}(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

Then define

$$\hat{l}(e,s) = \max\{x \mid \forall y < x\,(C_s(y) = \hat{\Phi}^{A_s}_{e,s}(y)\downarrow)\}$$

$$\hat{r}(e,s) = \max\{use\,\Phi^{A_s}_{e,s}(x) \mid x \leq \hat{l}(e,s)\}$$

Injury set: $\hat{I}_e = \{x \mid \exists s\,(x \in A_{s+1} - A_s \wedge x < \hat{r}(e,s))\}$

We enumerate x into A_{s+1}, only if $x \in B^{[e]}_{s+1}$ and $x \geq \hat{r}(i,s)$ for all $i \leq e$

[keep prior requirements not injured]

i.e. $A_{s+1} = A_s \cup \bigcup_{e \in \mathbb{N}} \{x \in B^{[e]}_{s+1} \mid \forall i \leq e\,(x > \hat{r}(i,s))\}$

Let $A = \bigcup_s A_s$

Now we need to prove (by induction on e) that

① $C \neq \Phi_e^A$

② $B^{[e]} \subseteq^* A^{[e]}$

Fix e, assume by induction that for all $i < e$, $C \neq \Phi_i^A$ and $A^{[i]} =^* B^{[i]}$.

Then $A^{[<e]} = \bigcup_{i<e} A^{[i]} =^* B^{[<e]}$ is rec.

Since $\hat{I}_e \subseteq A^{[<e]}$ (can only be injured by higher priority), so $\hat{I}_e \leq_T A^{[<e]}$

Hence $\hat{I}_e$ is also rec. [To decide if $n \in \hat{I}_e$ first ask if $n \in A^{[<e]}$; if not, say no, if so, wait until n comes into $A_{s+1} - A_s$, and see if $n < \hat{r}(e,s)$ or not]

<u>Claim 1</u> $C \neq \Phi_e^A$

Assume otherwise, i.e. $C = \Phi_e^A$. Then $\lim_s \hat{l}(e,s) = \infty$ [Since for each y, $\lim_s \hat{\Phi}_{e,s}^{A_s}(y) = \Phi_e^A(y) = C(y) = \lim_s C_s(y)$]

We show that $C \leq_T \hat{I}_e$

Given $p \in \mathbb{N}$, to compute $C(p)$,

Find the least s_p (effectively from p, with information from $\hat{I}_e$) s.t.

① $\hat{l}(e,s_p) > p$

② $\forall x \leq p \;\; \hat{I}_e \upharpoonright use\, \hat{\Phi}_{e,s_p}^{A_{s_p}}(x) \subseteq A_{s_p}$

Note that such s_p exists, actually there is s s.t. meet ① and $\forall x \leq p \; A \upharpoonright use\, \hat{\Phi}_{e,s}^{A_s}(x) \subseteq A_s$

But for each s, we can not effectively determined whether $A \upharpoonright use\, \hat{\Phi}_{e,s}^{A_s}(x) \subseteq A$ holds.

However, with information from $\hat{I}_e$, we can decide whether $\hat{I}_e \upharpoonright use\, \hat{\Phi}_{e,s}^{A_s}(x) \subseteq A_s$

[Given s, we can effectively compute $use\, \hat{\Phi}_{e,s}^{A_s}(x)$ and A_s and so $\hat{I}_e \upharpoonright use\, \hat{\Phi}_{e,s}^{A_s}(x)$]

By induction on $t \geq s_p$, we show

For all $t \geq s$ ① $\hat{\imath}(e,t) > p$

② $\hat{r}(e,t) \geq \max\{ \mathrm{use}\, \hat{\Phi}^{A_{s_p}}_{e,s_p}(x) \mid x \leq p \}$

Note that
① $\hat{\imath}(e,s_p) > p$ by the choice of s_p
② since $p < \hat{\imath}(e,s_p)$
$\hat{r}(e,s_p) = \max\{ \mathrm{use}\, \hat{\Phi}^{A_{s_p}}_{e,s_p}(x) \mid x \leq \hat{\imath}(e,s_p) \}$
$\geq \max\{ \mathrm{use}\, \hat{\Phi}^{A_{s_p}}_{e,s_p}(x) \mid x \leq p \}$
so, ①,② hold for s_p

Assume by induction that ①,② hold for all t' s.t. $s \leq t' < t$

Assume that ① fails for t, i.e. $p \geq \hat{\imath}(e,t)$, by the definition of $\hat{\imath}(e,t)$, there must be an $x \leq p$ s.t. $C_t(x) \neq \hat{\Phi}^{A_t}_{e,t}(x)$. choose x to be the least.

By induction hypothesis, we know $p < \hat{\imath}(e,t-1)$, and since $x \leq p$, we have $C_{t-1}(x) = \hat{\Phi}^{A_{t-1}}_{e,t-1}(x)$

By induction on $x \leq p$, we show:

For all $x \leq p$, we have: for all $t \geq s_p$

① $A_t \upharpoonright \mathrm{use}\, \hat{\Phi}^{A_t}_{e,t}(x) = A_{s_p} \upharpoonright \mathrm{use}\, \hat{\Phi}^{A_{s_p}}_{e,s_p}(x)$, ~~and~~

② $\hat{\Phi}^{A_t}_{e,t}(x)\downarrow = \hat{\Phi}^{A_{s_p}}_{e,s_p}(x)\downarrow$

Assume ①,② hold for all $z < x$. we prove ① also holds for x :

Assume not. Let t be the least s.t. $A_t \upharpoonright \mathrm{use}\, \hat{\Phi}^{A_t}_{e,t}(x) \neq A_{s_p} \upharpoonright \mathrm{use}\, \hat{\Phi}^{A_{s_p}}_{e,s_p}(x)$, then $t > s_p$ and for all t', s.t. $s_p \leq t' < t$, $A_{t'} \upharpoonright \mathrm{use}\, \hat{\Phi}^{A_{t'}}_{e,t'}(x) = A_{s_p} \upharpoonright \mathrm{use}\, \hat{\Phi}^{A_{s_p}}_{e,s_p}(x)$.

Since $\hat{\Phi}^{A_{s_p}}_{e,s_p}(x)\downarrow$ (because $x \leq p < \hat{\imath}(e,s_p)$), we have $\mathrm{use}\, \hat{\Phi}^{A_{t'}}_{e,t'}(x) = \mathrm{use}\, \hat{\Phi}^{A_{s_p}}_{e,s_p}(x)$. and also $\hat{\Phi}^{A_{t'}}_{e,t'}(x)\downarrow = \hat{\Phi}^{A_{s_p}}_{e,s_p}(x)\downarrow$

So there must be $y \in A_t - A_{t-1}$ s.t. $y < \mathrm{use}\, \hat{\Phi}^{A_{t-1}}_{e,t-1}(x) = \mathrm{use}\, \hat{\Phi}^{A_{s_p}}_{e,s_p}(x)$

Since $\hat{I}_e \upharpoonright \mathrm{use}\, \hat{\Phi}^{A_{s_p}}_{e,s_p}(x) \leq A_{s_p}$ and $y \notin A_{t-1} \supseteq A_{s_p}$, so either (i) $y \notin \hat{I}_e$ or (ii) $y \geq \mathrm{use}\, \hat{\Phi}^{A_{s_p}}_{e,s_p}(x)$

Case (i) $y \geq \hat{r}(e,t-1) = \max\{ \mathrm{use}\, \hat{\Phi}^{A_{t-1}}_{e,t-1}(z) \mid z \leq \hat{\imath}(e,t-1) \}$

By IH②, for all $z < x$, $\hat{\Phi}^{A_s}_{e,s}(z)\downarrow = \hat{\Phi}^{A_{s_p}}_{e,s_p}(z)\downarrow$ for all $s \geq s_p$. therefore

$C(z) = \Phi^{A}_e(z) = \lim_s \hat{\Phi}^{A_s}_{e,s}(z) = \hat{\Phi}^{A_{s_p}}_{e,s_p}(z) = C_{s_p}(z)$ since $z < x \leq p < \hat{\imath}(e,s_p)$

Hence $C_{t-1}(z) = C_{s_p}(z) = \hat{\Phi}^{A_{t-1}}_{e,t-1}(z)$ (since $C_s(z)$ can change only once)

Thus $x \leq \hat{\imath}(e,t-1)$, so $y \geq \mathrm{use}\, \hat{\Phi}^{A_{t-1}}_{e,t-1}(x)$ $\rightarrow\leftarrow$

Case (ii) $y \geq \mathrm{use}\, \hat{\Phi}^{A_{s_p}}_{e,s_p}(x) = \mathrm{use}\, \hat{\Phi}^{A_{t-1}}_{e,t-1}(x)$ $\rightarrow\leftarrow$

To see ② holds for x : we have $\hat{\Phi}^{A_{t-1}}_{e,t-1}(x)\downarrow = \hat{\Phi}^{A_{s_p}}_{e,s_p}(x)\downarrow$ and $A_t \upharpoonright \mathrm{use}\, \hat{\Phi}^{A_t}_{e,t}(x) = A_{s_p} \upharpoonright \mathrm{use}\, \hat{\Phi}^{A_t}_{e,t}(x) = A_{t-1} \upharpoonright \mathrm{use}\, \hat{\Phi}^{A_{t-1}}_{e,t-1}$

so $\hat{\Phi}^{A_t}_{e,t}(x)\downarrow = \hat{\Phi}^{A_{s_p}}_{e,s_p}(x)\downarrow$.

After all, we have $C(p) = \Phi^{A}_e(p) = \lim_s \hat{\Phi}^{A_s}_{e,s}(p) = \hat{\Phi}^{A_{s_p}}_{e,s_p}(p)$. so $C \leq_T \hat{I}_e$ $\rightarrow\leftarrow$ $\square$ claim 1.

Claim 2 Let $T = \{ s \mid A_s \upharpoonright a_s = A \upharpoonright a_s \}$ (the set of true stage in the enumeration of A)

Then $\lim_{t \in T} \hat{r}(e,t) < \infty$

By claim 1, we have $C \neq \Phi_e^A$. Let $p \in \mathbb{N}$ be the least s.t. $C(p) \neq \Phi_e^A(p)$.

So $C(x) = \Phi_e^A(x)$ for all $x < p$.

$= \lim_s \hat{\Phi}_{e,s}^{A_s}(x)$

Let s be the least s.t. ① $C_s(x) = C(x)$ for all $x \leq p$

② $A \upharpoonright \text{use}\,\Phi_e^A(x) = A_s \upharpoonright \text{use}\,\Phi_e^A(x)$ and $\Phi_e^A(x) = \hat{\Phi}_{e,s}^{A,s}(x)\downarrow$ for all $x < p$

Case 1: $\hat{\Phi}_{e,t}^{A_t}(p)\uparrow$ for all $t \geq s$ and $t \in T$

[For all $t \geq s$, $a_t > \text{use}\,\Phi_e^{A}(x) = \text{use}\,\hat{\Phi}_{e,s}^{A_s}(x)$ so $\hat{\Phi}_{e,t}^{A_t}(x) = \Phi_{e,t}^{A_t}(x)$]

~~Then $\hat{l}(e,t)$~~

Note: for all $t \geq s$, $C_t(x) = C(x) = \Phi_e^A(x) = \hat{\Phi}_{e,t}^{A_t}(x)\downarrow$

Thus $\hat{l}(e,t) = \hat{l}(e,s) = p$ for all $t \geq s$.

Hence $\hat{r}(e,t) = \max\{ \text{use}\,\hat{\Phi}_{e,t}^{A_t}(x) \mid x \leq \hat{l}(e,t) \}$

$= \max\{ \text{use}\,\hat{\Phi}_{e,t}^{A_t}(x) \mid x \leq p \}$

$= \max\{ \text{use}\,\hat{\Phi}_{e,s}^{A_s}(x) \mid x \leq p \}$ for all $t \geq s$, which is fixed.

Case 2: $\hat{\Phi}_{e,t}^{A_t}(p)\downarrow$ for some $t \geq s$ and $t \in T$.

Note that if $t \in T$, i.e. $A_t \upharpoonright a_t = A \upharpoonright a_t$ and $\hat{\Phi}_{e,t}^{A_t}(p)\downarrow = y$,

then $\text{use}\,\hat{\Phi}_{e,t}^{A_t}(p) = \text{use}\,\Phi_{e,t}^{A_t}(p) < a_t$ so $A_t \upharpoonright \text{use}\,\Phi_{e,t}^{A_t}(p) = A \upharpoonright \text{use}\,\Phi_{e,t}^{A_t}(p)$.

So $\hat{\Phi}_{e,t}^{A_t}(p)\downarrow = \Phi_{e,t}^{A_t}(p) = \Phi_e^A(p) \neq C(p)$ for all such t.

Hence $\hat{l}(e,t) = p$ for all such t,

and $\hat{r}(e,t) = \max(\{ \text{use}\,\hat{\Phi}_{e,s}^{A_s}(x) \mid x < p \} \cup \{ \text{use}\,\hat{\Phi}_{e,t}^{A_t}(p) \})$ for all t, which is also fixed.

$\square$ claim 2

Claim 3 $B^{[e]} \subseteq^* A^{[e]}$ for all $e \in \mathbb{N}$.

subclaim: T is infinite.

Suppose not, there is $s \in \mathbb{N}$ s.t. for all $t \geq s$, $A_t \upharpoonright a_t \subsetneq A \upharpoonright a_t$.

Let $s_0 = s$, let s_1 be the least $> s_0$ s.t. there is $x \in A_{s_1} - A_{s_1 - 1}$ s.t. $x < a_{s_0}$, so $a_{s_1} \leq x < a_{s_0}$

in this way, we can find $a_{s_0} > a_{s_1} > a_{s_2} \cdots$ $\rightarrowtail\!\leftarrow$

Let $\hat{r}(e) = \lim_{t \in T} \hat{r}(e,t)$. By our genericity of A, for all $x > \hat{r}(e)$, $x \in B^{[e]} \Rightarrow x \in A^{[e]}$

$\square$