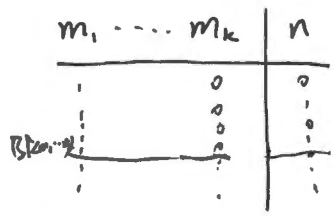
Truth table reduction

- We say $A \le_{tt} B$ iff there is a rec. function h s.t. for each $n \in \mathbb{N}$

  $h(n)$ is a truth-table :



and $n \in A$ iff to the right of the line $B\restriction\langle m_1 \cdots m_k\rangle$ is 1

or we put it this way : $A \le_{tt} B$ iff there is rec. function $M : \mathbb{N} \to \mathbb{N}^{<\omega}$ and $T : \mathbb{N} \to \bigcup_{v \in \omega} 2^{(2^v)}$

s.t. $\Big\}$ If $\underline{M(n) \in \mathbb{N}^v}$ Then $\underline{T(n) \in 2^{(2^v)}}$ And. ~~for each~~ $n \in A$ iff $T(n)(B\restriction M(n))=$

for each $x \in \mathbb{N}$ $M(n)$ is a sequence for each $v \in \mathbb{N}$ of number of length v, $T(n)$ is a ~~truth~~ v-ary Boolean function

Many-one reduction is ~~a~~ the special case of truth-table reduction

if $A \le_m B$ ⊕, and f witness $A \le_m B$. ~~Then~~ Let M, T be s.t. for each n, $M(n) = \langle f(n) \rangle$, $T(n) = \frac{1}{0}\Big|\frac{1}{0}$

Then M, T witness that $A \le_{tt} B$.

So tt-reduction is weaker than many-one reduction

Bounded truth-table reduction

We say $A \le_{btt} B$ if there is a number $b \in \mathbb{N}$ and rec. function $M : \mathbb{N} \to \mathbb{N}^b$,

$T : \mathbb{N} \to \underbrace{2^{2^b}}_{\text{finitely many}}$ s.t. for each $n \in \mathbb{N}$. $n \in A$ iff $T(n)(B\restriction M(n)) = 1$

Theorem If C is creative, S is simple, then $C \not\le_{btt} S$.

Fact $A \le_{btt} B$ iff ~~$A \le_{tt} B$ and~~ T witness $A \le_{tt} B$, and $T[\mathbb{N}]$ is finite.

Proof of $C \not\leq_{btt} S$

Assume function M, T, witness that $C \leq_{btt} S$ and $\{T_1 \cdots T_k\}$ contains all the truth table involved in the reduction. Let p witness that C is creative.

We conclude a contradition by showing that for each $a \in \mathbb{N}$, there is a program Π_a s.t. for each $e \in \mathbb{N}$ if $W_e \subseteq \bar{C}$, $\Pi_a(e) = (n, m, t, I)$ s.t. $[\, n \notin C \cup W_e$, $m = M(n)$, $t = T(n)$, (so assume t is a ν-ary ~~truth table~~ Boolean function and the length of m is ν), and I is a sequence of length a: $\langle i_1, \cdots i_a \rangle$ s.t. $1 \leq i_1 < \cdots < i_a \leq \nu$ and for each $1 \leq k \leq a$, $m_{i_k} \in S \,]$

Note: $a \leq \nu$,

If such Π_a exists for each a, then we have arbitrary large truth-table involved.

so, it comes to a contradiction.

Now, we prove that Π_a exists by induction on a,

Π_0 exists: Π_0 can be programmed: Given e, let $n = p(e)$, $m = M(n)$, $t = T(n)$, and $I = \langle\, \rangle$

Assume Π_a exists, we construct Π_{a+1}.

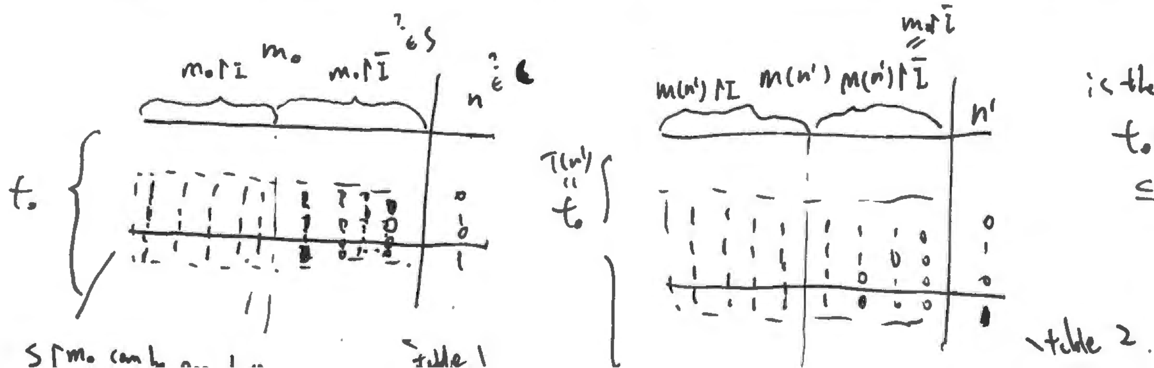
Fix e, first, let $s_0 = \Pi_a(e) = (n_0, m_0, t_0, I_0)$ Note that $|I_0| = a$, $m_0 \upharpoonright I_0 \subseteq S$, $n_0 \notin C \cup W_e$

We construct a r.e. set $W_{e'}$ by first generating W_e, and put n_0 in it. Then we find each n' s.t. $T(n') = t_0$, $M(n') \upharpoonright \bar{I}_0 = m_0 \upharpoonright \bar{I}_0$, $M(n') \upharpoonright I_0 \subseteq S$

And put it in $W_{e'}$ so

$$W_{e'} = W_e \cup \{\underset{\Pi_a(e)(0)}{n_0}\} \cup \{ n' \in \mathbb{N} \mid T(n') = t_0 = \Pi_a(e)(2) \text{ and } M(n') \upharpoonright \bar{I}_0 = \overset{\Pi_a(e)(1)}{m_0} \upharpoonright \overset{\Pi_a(e)(3)}{\bar{I}_0} \text{ and } M(n') \upharpoonright I_0 \subseteq S \}$$

Note that $W_{e'}$ is r.e. ~~and~~ $W_e \subsetneq W_{e'}$ and $W_{e'} \subseteq \bar{C}$ [Since table 1 tells $n \notin C$, and table 2 is the same with table 1, in the sense $t_0 = T(n')$, both $m_0 \upharpoonright I$, $M(n') \upharpoonright I \subseteq S$, and $m_0 \upharpoonright \bar{I} = M(n') \upharpoonright \bar{I}$



Note also that, with rec function M, T, ~~fixed~~ fixed, Π_a given, there is a rec function g, s.t. $W_{e'} = W_{g(e)}$

So we obtain the sequence: $W_e = W_{g^{(0)}(e)} \subsetneq W_{g^{(1)}(e)} \subsetneq W_{g^{(2)}(e)} \cdots$

And for each $g^{(j)}(e)$, Let $\delta_j = \Pi_a(g^{(j)}(e)) = (n_j, m_j, t_j, I_j)$

So $\Sigma = \{\delta_j \mid j \in \mathbb{N}\}$ is an infinite r.e. set (note that $n_{j_1} \neq n_{j_2}$ if $j_1 \neq j_2$)

Claim $\Sigma' = \{ m \restriction \bar{I} \mid \exists n, t \ (n, m, t, I) \in \Sigma \}$ is also infinite and r.e.

It is clearly r.e. to see it is infinite, note that there are only finite many pairs (t_j, I_j) occurs in Σ, so there is (t, I), such that there are infinitely many δ_j in Σ has the form (n_j, m_j, t, I).

Let $j_1 < j_2$ s.t. $\delta_{j_1} = (n_{j_1}, m_{j_1}, t, I)$, $\delta_{j_2} = (n_{j_2}, m_{j_2}, t, I)$.

Now further assume $m_{j_1} \restriction \bar{I} = m_{j_2} \restriction \bar{I}$, then

$n_{j_2} \in W_{g^{(j_1)}(e)} \cup \{n_{j_1}\} \cup \{ n' \in \mathbb{N} \mid T(n') = t \text{ and } \underset{}{m(n') \restriction \bar{I}} = \overset{m_{j_2} \restriction \bar{I}}{m_{j_1} \restriction \bar{I}} \text{ and } m(n') \restriction I \in S \}$

$= W_{g^{(j_1+1)}(e)}$

But $n_{j_2} \notin W_{g^{(j_2)}(e)}$ and $W_{g^{(j_1+1)}(e)} \subseteq W_{g^{(j_2)}(e)}$ $\vdash \!\!\times$

Therefore $m_{j_1} \restriction \bar{I} \neq m_{j_2} \restriction \bar{I}$, and So Σ' is infinite.

Note the $\bigcup \Sigma'$ is also an infinite r.e. set.

so $\bigcup \Sigma' \cap S \neq \emptyset$. Let b be the first we find in the r.e. set $\bigcup \Sigma' \cap S$.

and δ be the first in Σ contain b in its first coordinate, ~~and its index is~~

$\delta_j = (n_j, m_j, t_j, I_j)$

Let $\Pi_{a+1}(e) = (n_j, m_j, t_j, I_j')$ where the index of b in $m_j = i$

and $I_j' = I_j \cup \{i\}$.

$\square$